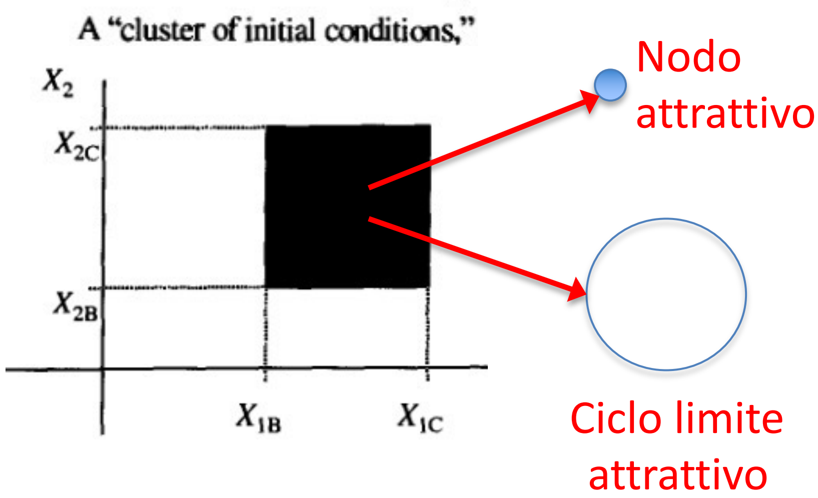
$$\dot{X}_1 = f_1(X_1, X_2)$$
$$\dot{X}_2 = f_2(X_1, X_2)$$

Flussi dissipativi in due dimensioni

$$\frac{1}{A}\frac{dA}{dt} = \frac{\partial f_1}{\partial X_1} + \frac{\partial f_2}{\partial X_2} < 0$$

fixed points (dim.0)

limit cycles (dim.1)



Metodo dello Jacobiano per studiare i punti fissi nel caso generale a 2 dim.

Equazioni originarie

$$\dot{X}_1 = f_1(X_1, X_2)$$
$$\dot{X}_2 = f_2(X_1, X_2)$$

…ricavare i punti fissi…

Equazioni linearizzate nelle vicinanze di un dato punto fisso (X_{1o}, X_{2o})

$$\dot{x}_1 = \frac{\partial f_1}{\partial x_1} x_1 + \frac{\partial f_1}{\partial x_2} x_2$$
$$\dot{x}_2 = \frac{\partial f_2}{\partial x_1} x_1 + \frac{\partial f_2}{\partial x_2} x_2$$

with $f_{ij} = \dfrac{\partial f_i}{\partial x_j}$ …calcolate nel punto fisso

Distanze dal punto fisso

$$x_1 = X_1 - X_{1o}$$
$$x_2 = X_2 - X_{2o}$$

3.14 The Jacobian Matrix for Characteristic Values

We would now like to introduce a more elegant and general method of finding the characteristic equation for a fixed point. This method makes use of the so-called ***Jacobian matrix*** of the derivatives of the time evolution functions. Once we see how this procedure works, it will be easy to generalize the method, at least in principle, to find characteristic values for fixed points in state spaces of any dimension. The Jacobian matrix for the system is defined to be the following square array of the derivatives:

Matrice Jacobiana

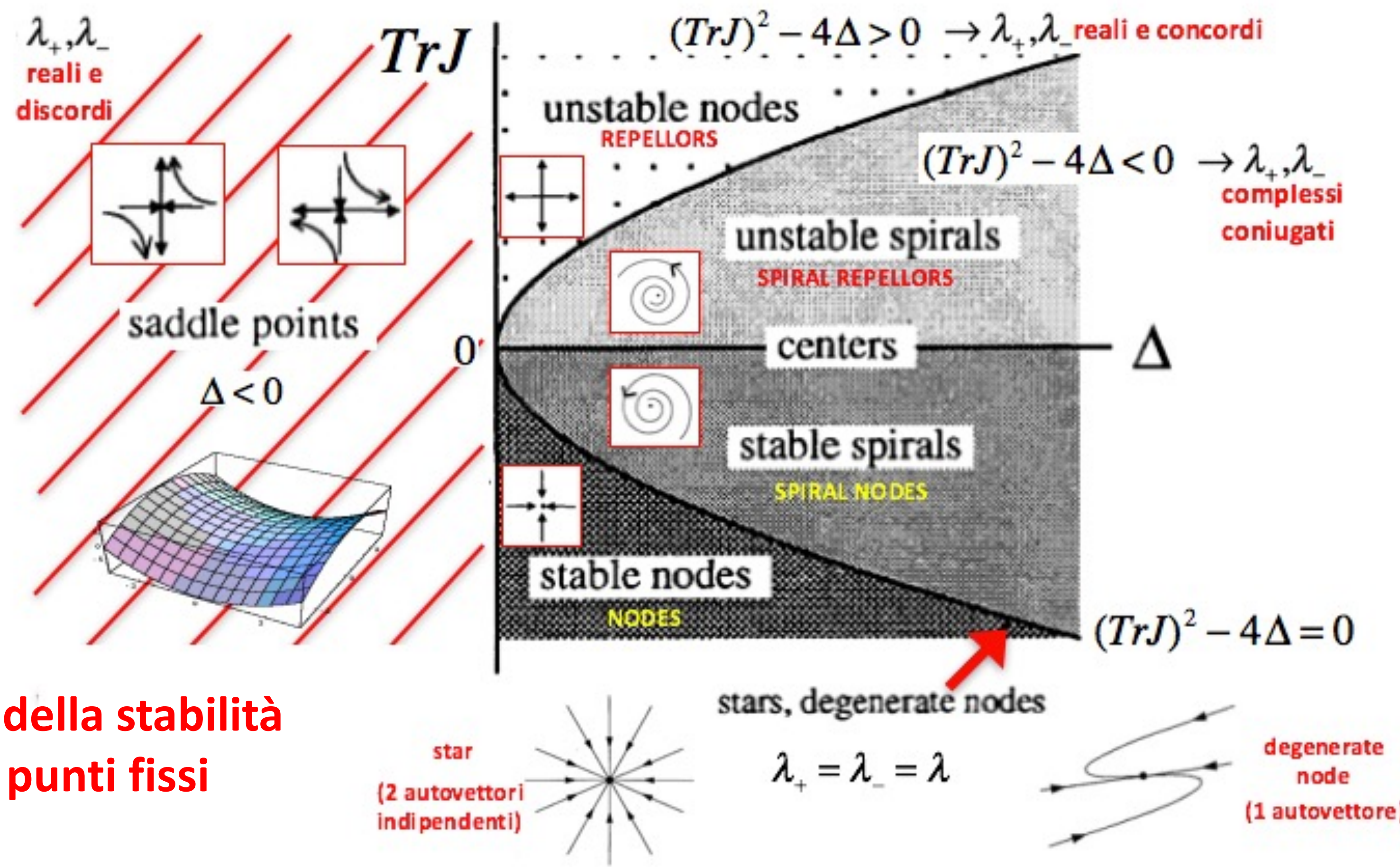
$$J = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \qquad (3.14\text{-}1)$$

Autovalori: λ_+, λ_-

where the derivatives are evaluated at the fixed point. We subtract λ from each of the principal diagonal (upper left to lower right) elements and set the determinant of the matrix equal to 0:

3.18 Summary

In this chapter we have developed much of the mathematical machinery needed to discuss the behavior of dynamical systems. We have seen that fixed points and their characteristic values (determined by derivatives of the functions describing the dynamics of the system) are crucial for understanding the dynamics. We have also seen that the dimensionality of the state space plays a major role in determining the kinds of trajectories that can occur for bounded systems.



Studio della stabilità dei punti fissi

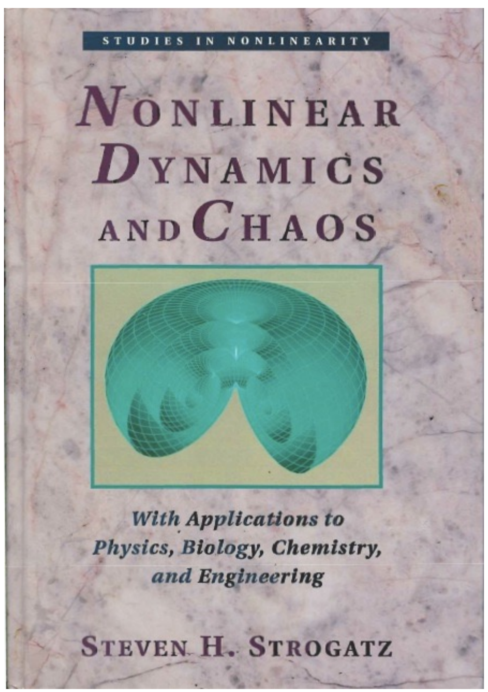
Steven
Strogatz

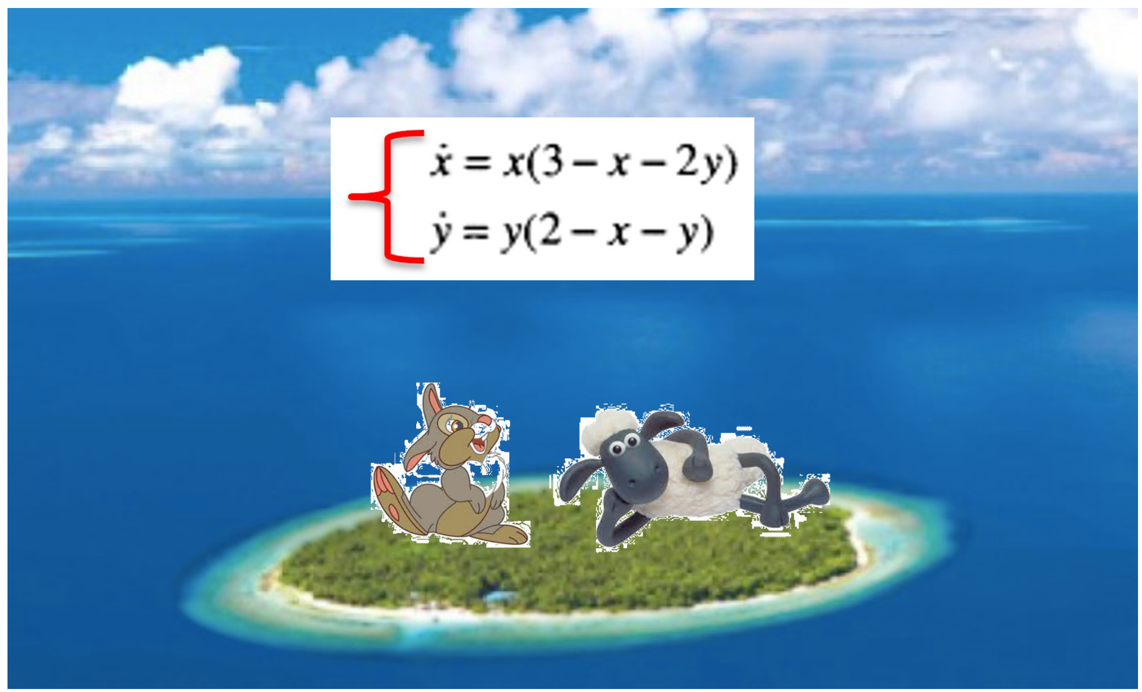
Sheep

Rabbit

6.4 Rabbits versus Sheep

In the next few sections we'll consider some simple examples of phase plane analysis. We begin with the classic ***Lotka–Volterra model of competition*** between two species, here imagined to be rabbits and sheep. Suppose that both species are competing for the same food supply (grass) and the amount available is limited. Furthermore, ignore all other complications, like predators, seasonal effects, and other sources of food. Then there are two main effects we should consider:





$$\begin{cases} \dot{x} = x(3 - x - 2y) \\ \dot{y} = y(2 - x - y) \end{cases}$$

Steven
Strogatz

Sheep

Rabbit

6.4 Rabbits versus Sheep

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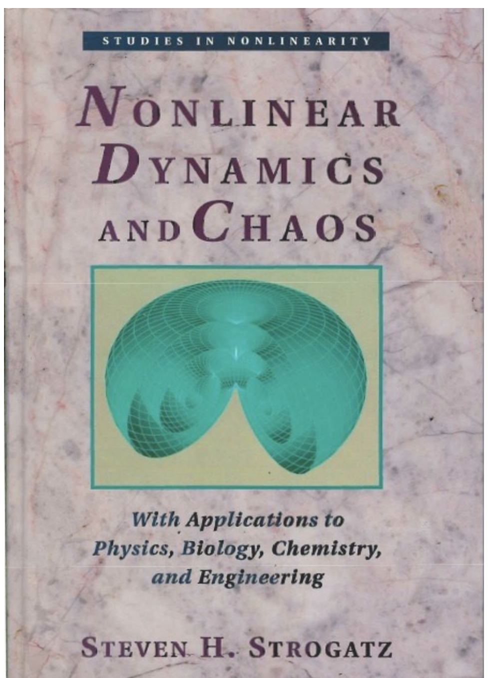


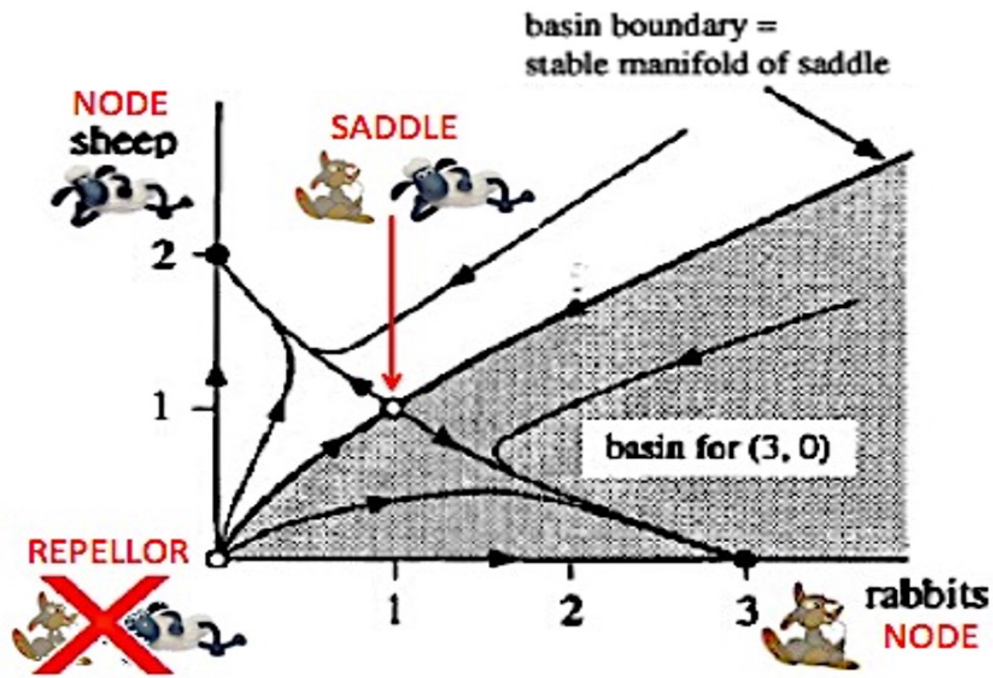
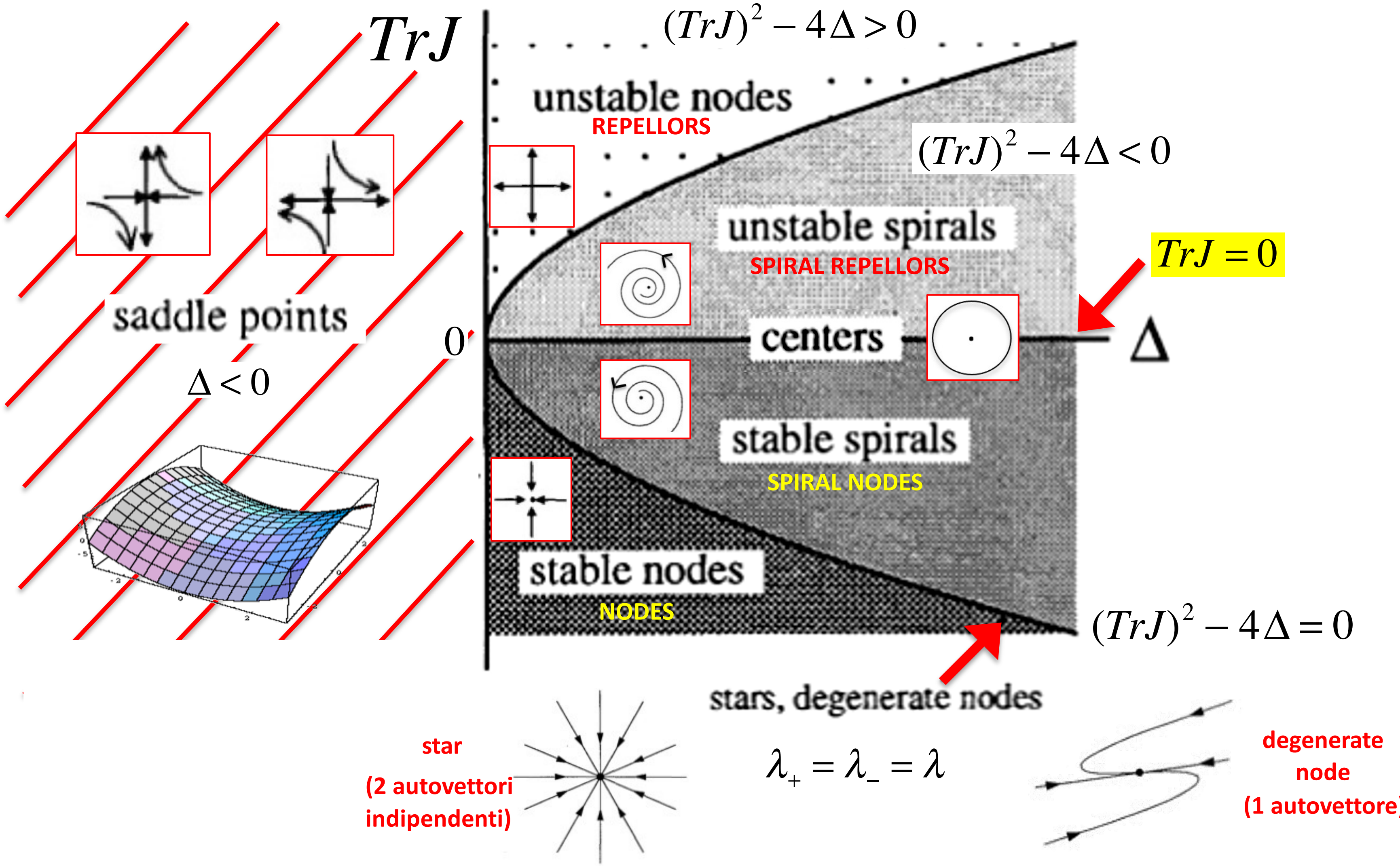


Diagramma dei Punti Fissi in uno Spazio degli Stati a Due Dimensioni

$$\lambda_{\pm} = \frac{TrJ \pm \sqrt{(TrJ)^2 - 4\Delta}}{2}$$



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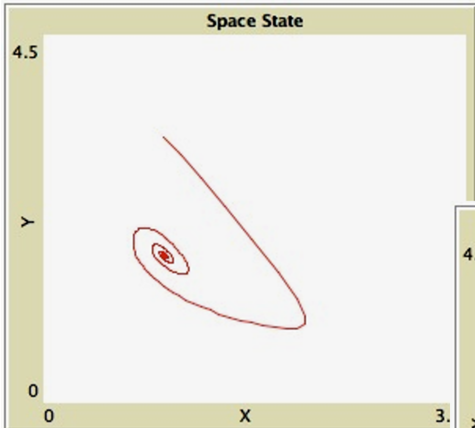
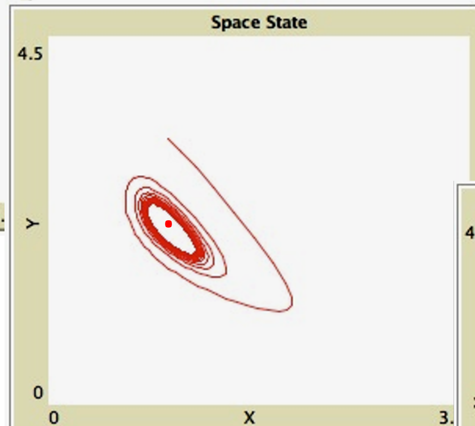
A=1, B=1.80



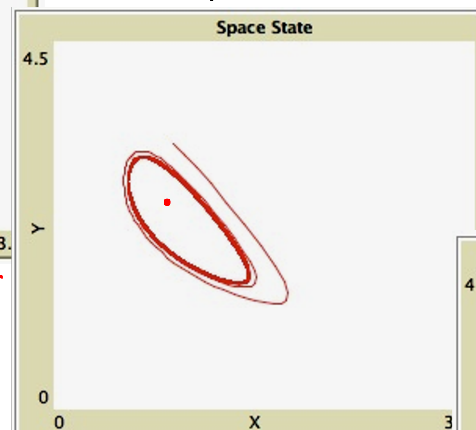
1 stable spiral node

A=1, B=2.15



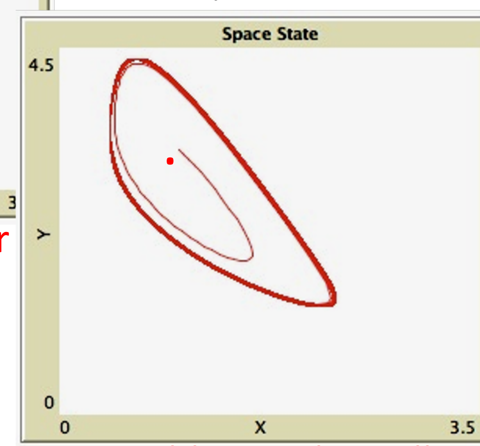
1 unstable spiral repellor
+
1 limit cycle

A=1, B=2.33



1 unstable spiral repellor
+
1 limit cycle
1

A=1, B=2.85



1 unstable spiral repellor
+
1 limit cycle

$$\dot{X} = A - (B+1)X + X^2 Y$$
$$\dot{Y} = BX - X^2 Y$$

1 punto fisso:

$$X_0 = A,\ Y_0 = B/A.$$

A=1,B<2 : stable spiral node

A=1, B=2 : Nasce un ciclo limite!

A=1, B>2 :
unstable spiral repellor + 1 limit cycle

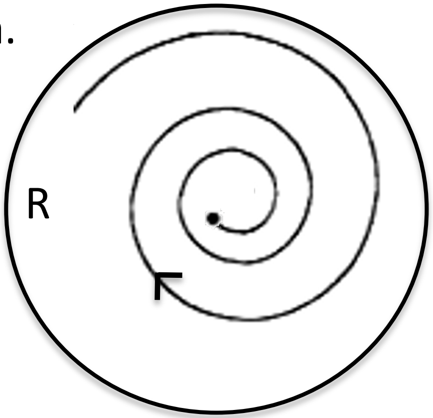
Il Teorema di Poincaré-Bendixson

We shall formulate the analysis in answer to two questions: (1) When do limit cycles occur? and (2) When is a limit cycle stable or unstable? The first question is answered for a two-dimension state space by the famous ***Poincaré–Bendixson Theorem***. The theorem can be formulated in the following way:

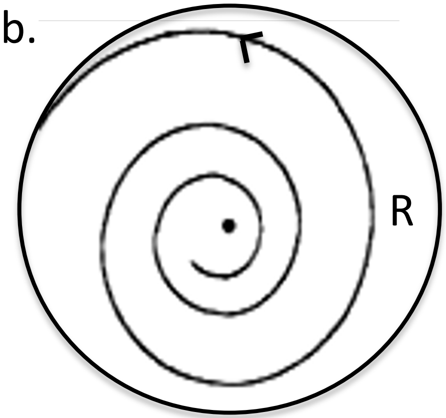
1. Suppose the long-term motion of a state point in a two-dimensional state space is limited to some finite-size region; that is, the system doesn't wander off to infinity.
2. Suppose that this region (call it R) is such that any trajectory starting within R stays within R for all time. [R is called an "invariant set" for that system.]
3. Consider a particular trajectory starting in R. The Poincaré–Bendixson Theorem states that there are only two possibilities for that trajectory:

a. The trajectory approaches a fixed point of the system as $t \to \infty$.
b. The trajectory approaches a limit cycle as $t \to \infty$.

a.

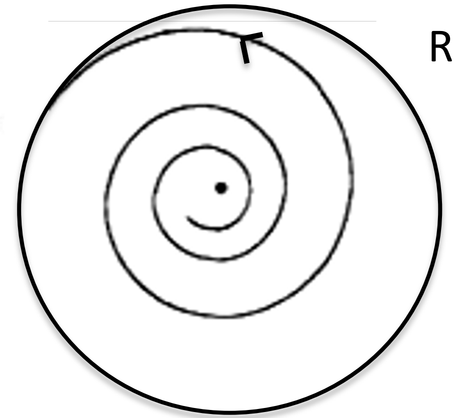


b.



Il Teorema di Poincaré-Bendixson

A proof of this theorem is beyond the scope of this book. The interested reader is referred to [Hirsch and Smale, 1974]. We can see, however, that the results are entirely reasonable if we take into account the No-Intersection Theorem and the assumption of a bounded region of state space in which the trajectories live. The reader is urged to draw some pictures of state space trajectories in two dimensions to see that these two principles guarantee that the only two possibilities are fixed points and limit cycles.

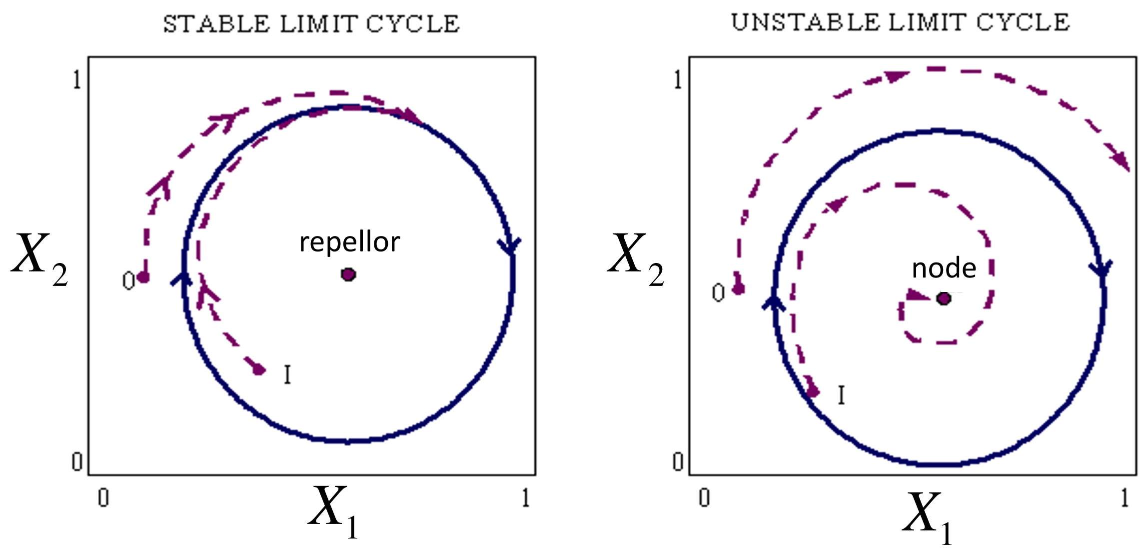


It is important to note that the Poincaré–Bendixson Theorem works only in two dimensions because only in two dimensions does a closed curve separate the space into a region "inside" the curve and a region "outside." Thus a trajectory starting inside the limit cycle can never get out and a trajectory starting outside can never get in. This argument is an excellent example of the power of topological arguments in the study of dynamical systems. Further, from the Poincaré–Bendixson Theorem we arrive at an important result: Chaotic trajectories (in a bounded system) cannot occur in a state space of two dimensions. *For systems described by differential equations, we need at least three state-space dimensions for chaos.*

3.16 Poincaré Sections and the Stability of Limit Cycles

We have seen that in state spaces of two (or more) dimensions, a new type of behavior can arise: motion on a limit cycle. The obvious question is the following: Is the motion on the limit cycle stable? That is, if we push the system slightly away from the limit cycle, does it return to the limit cycle (at least asymptotically) or is it repelled from the limit cycle? As we shall see, both possibilities occur in actual systems.

You might expect that we would proceed much as we did for nodes and repellors, by calculating characteristic values involving derivatives of the functions describing the state space evolution. In principle, one could do this, but Poincaré showed that an algebraically and conceptually much simpler method suffices. This method uses what is called a ***Poincaré section*** of the limit cycle. The Poincaré section is closely related to the stroboscopic portraits used in Chapter 1 to discuss the behavior of the diode circuit.



Costruzione della Sezione di Poincaré

For a two-dimensional state space, the Poincaré section is constructed as follows. In the two-dimensional state space, we draw a line segment that cuts through the limit cycle as shown in Fig. 3.12 (a). This line can be any line segment, but in some cases one might wish to choose the X_1 or X_2 axes. Let us call the point at which the limit cycle crosses the line segment going, say, point P.

J.H.Poncaré (1854-1912)

(a)

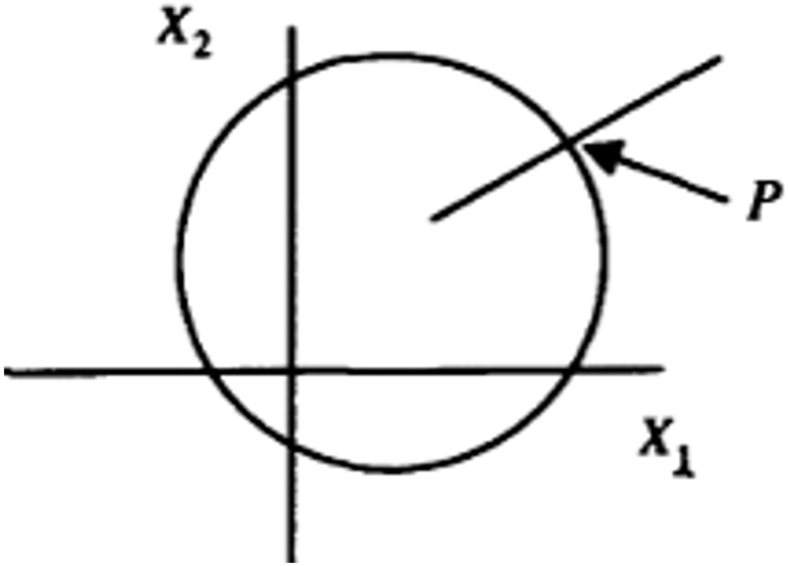
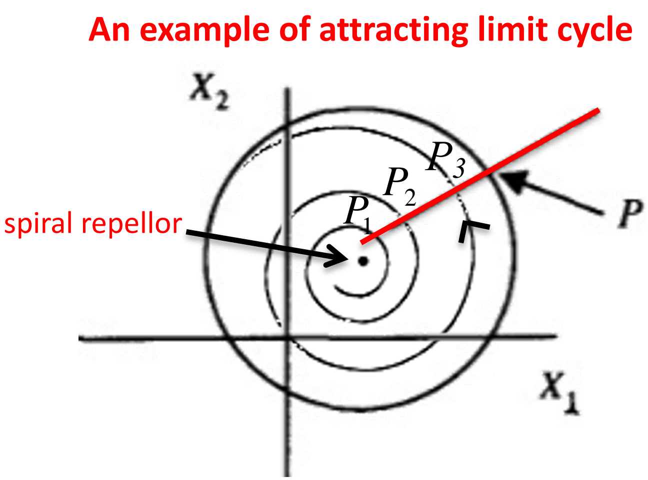


Fig. 3.12. (a) The Poincaré line segment intersects the limit cycle at point P. (b) The four possibilities for sequences of Poincaré intersection points for trajectories near a limit cycle in two dimensions.

If we now start a trajectory in the state space at a point that is close to, but not on, the limit cycle, then that trajectory will cross the Poincaré section line segment at a point other than P. Let's call the first crossing point P_1. As the trajectory evolves, it will cross the Poincaré line segment again at points P_2, P_3, and so on. If the sequence of points approaches P as time goes on for any starting point in the neighborhood of the limit cycle, we say that we have an ***attracting limit cycle*** or, equivalently, a ***stable limit cycle***. In other words, the limit cycle is an attractor for the system. If the sequence of intersection points moves away from P (for any trajectory starting near the limit cycle), we say we have a ***repelling limit cycle*** or, equivalently, an ***unstable limit cycle***. Another possibility is that the points are attracted on one side and repelled on the other: In that case we say that we have a ***saddle cycle*** (in analogy with a saddle point). These possibilities are shown graphically in Fig. 3.12 (b).



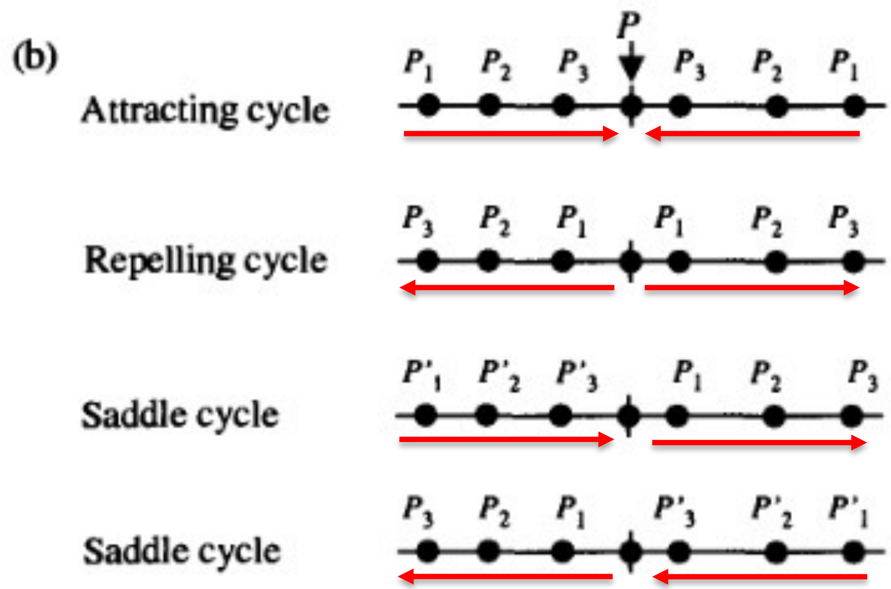


Fig. 3.12. (a) The Poincaré line segment intersects the limit cycle at point P. (b) The four possibilities for sequences of Poincaré intersection points for trajectories near a limit cycle in two dimensions.

How do we describe these properties quantitatively? We use what is called a ***Poincaré map function*** (or ***Poincaré map***, for short). The essential idea is that given a point P_1, where a trajectory crosses the Poincaré line segment, we can in principle determine the next crossing point P_2 by integrating the time-evolution equations describing the system. So, there must be some mathematical function, call it F, that relates P_1 to P_2: $P_2 = F(P_1)$. (Of course, finding this function F is equivalent to solving the original set of equations and that may be difficult or impossible in actual practice.) In general, we may write

$$P_{n+1} = F(P_n) \tag{3.16-1}$$

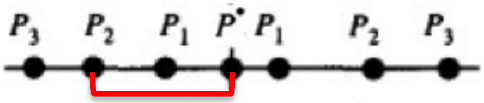
In general the function F depends not only on the original equations describing the system, but on the choice of the Poincaré line segment as well.

To analyze the nature of the limit cycle, we can analyze the nature of the function F and its derivatives. Two points are important to notice:

1. The Poincaré section reduces the original two-dimensional problem to a one-dimensional problem.
2. The Poincaré map function states an iterative (finite-size time step) relation rather than a differential (infinitesimal time step) relation.

The last point is important because F gives P_{n+1} in terms of P_n. The time interval between these points is roughly the time to go around the limit cycle once, a relatively big jump in time. On the other hand, a one-dimensional differential equation $\dot{x} = f(x)$ tells us how x changes over an infinitesimal time interval. The function F is sometimes called an *iterated map function* (or *iterated map*, for short). (Because of the importance of iterated maps in nonlinear dynamics, we shall devote Chapter 5 to a study of their properties.)

Let us note that the point P on the limit cycle satisfies $P = F(P)$. Any point P^* that satisfies $P^* = F(P^*)$ is called a *fixed point* of the map function. If a trajectory crosses the line segment exactly at P^*, it returns to P^* on every cycle. In analogy with our discussion of fixed points for differential equations, we can ask what happens to a point P_1 close to P^*. In particular, we ask what happens to the distance between P_1 and P^* as the system evolves. Formally, we look at



$$P_2 - P^* = F(P_1) - F(P^*) \tag{3.16-2}$$

and use a Taylor series expansion about the point P^* to write

$$P_2 - P^* = F(P^*) + \left.\frac{dF}{dP}\right|_{P^*} (P_1 - P^*) + \ldots - F(P^*) \tag{3.16-3}$$

If we define $d_i = (P_i - P^*)$, we see that

$$d_2 = \left.\frac{dF}{dP}\right|_{P^*} d_1 \tag{3.16-4}$$

We now define the ***characteristic multiplier M*** for the Poincaré map:

$$M = \left.\frac{dF}{dP}\right|_{P^*} \qquad \textbf{(M>0)} \tag{3.16-5}$$

M is also called the ***Floquet multipler*** or the ***Lyapunov multiplier***. In terms of M, we can write Eq. (3.16-4)

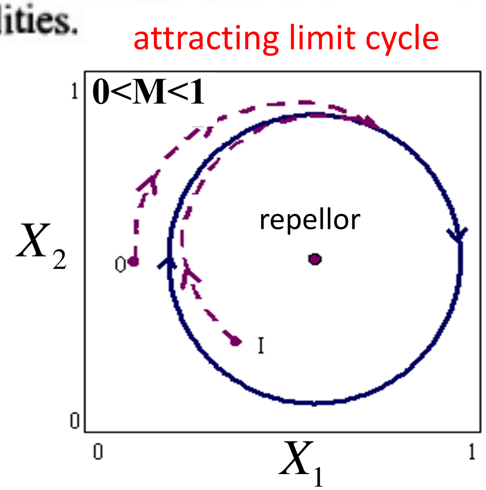
$$d_2 = Md_1 \tag{3.16-6}$$

We find in general

$$d_{n+1} = M^n d_1 \tag{3.16-7}$$

$$d_{n+1} = M^n d_1 \quad (3.16\text{-}7)$$

We see that if $M < 1$, then $d_2 < d_1$, $d_3 < d_2$, and so on: The intersection points approach the fixed point P. In that case the cycle is an attracting limit cycle. If $M > 1$, then the distances grow with repeated iterations, and the limit cycle is a repelling cycle. For saddle cycles, M is equal to 1 but the derivative of the map function is greater than 1 on one side of the cycle and less than 1 on the other side. However, based on our discussion of saddle points for one-dimensional state spaces, we expect that saddle cycles are rare in two-dimensional state spaces. Table 3.4 lists the possibilities.



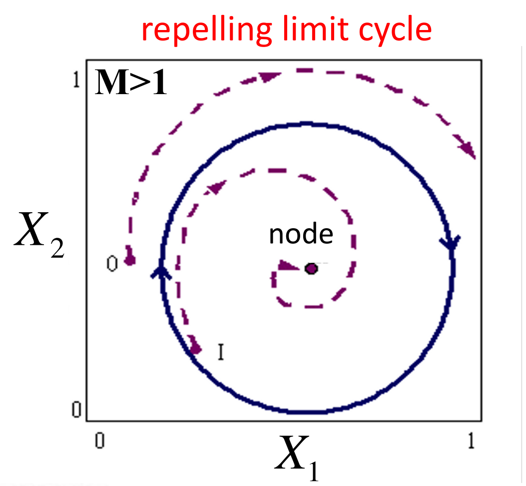


Table 3.4.

The Possible Limit Cycles and Their Characteristic Multipliers for Two–Dimensional State Space

Characteristic Multiplier	Type of Cycle
M < 1	Attracting Cycle
M > 1	Repelling Cycle
M = 1	Saddle Cycle (rare in two-dimensions)

We can also define a ***characteristic exponent*** associated with the cycle by the equation

$$M \equiv e^{\lambda} \tag{3.16-8}$$

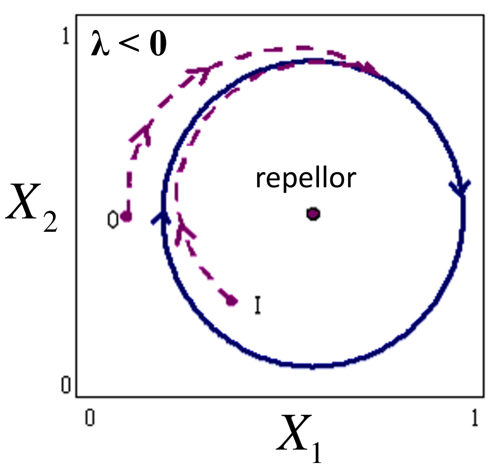
or

$$\lambda \equiv \ln(M) \tag{3.16-9}$$

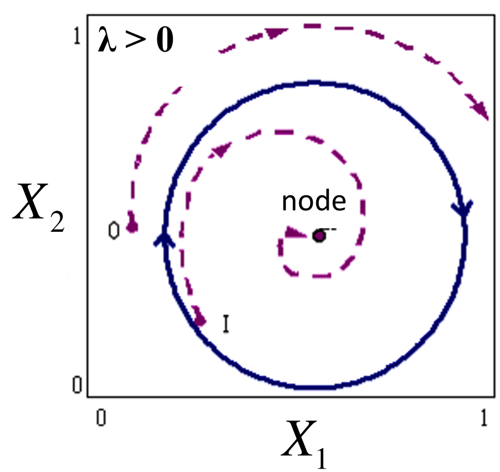
The idea is that the characteristic exponent plays the role of the Lyapunov exponent but the time unit is taken to be the time from one crossing of the Poincaré section to the next.

Let us summarize: The Poincaré section method allows us to characterize the possible types of limit cycles and to recognize the kinds of changes that take place in those limit cycles. However, in most cases, we cannot find the mapping function F explicitly; therefore, our ability to predict the kinds of limit cycles that occur for a given system is limited.

attracting limit cycle



repelling limit cycle



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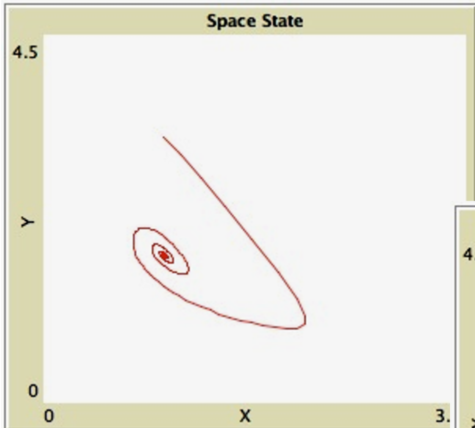
$$\dot{X} = A-(B+1)X + X^2Y$$
$$\dot{Y} = BX - X^2Y$$

1 punto fisso:

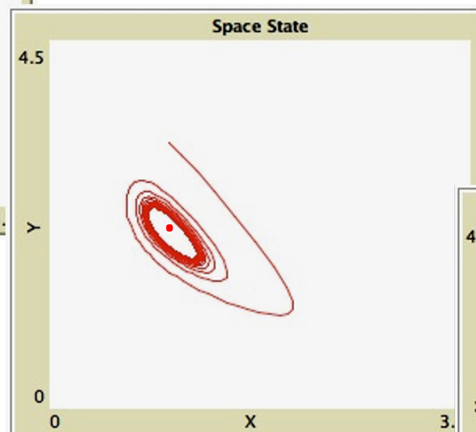
$$X_0 = A,\ Y_0 = B/A.$$

A=1, B=1.80



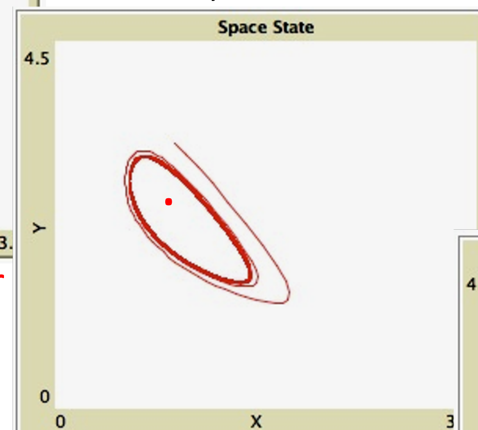
1 stable spiral node

A=1, B=2.15



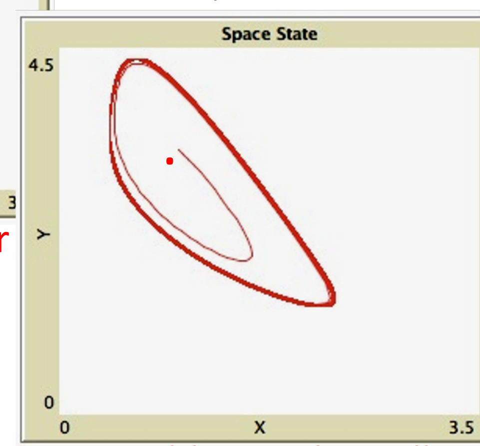
1 unstable spiral repellor
+
1 **stable** limit cycle

A=1, B=2.33



1 unstable spiral repellor
+
1 **stable** limit cycle

A=1, B=2.85



1 unstable spiral repellor
+
1 **stable** limit cycle

A=1,B<2 : stable spiral node

A=1, B=2 : Nasce un ciclo limite stabile!

A=1, B>2 :
unstable spiral repellor + 1 **stable** limit cycle

Ex.1 ROMEO E GIULIETTA

Il libro di Strogatz suggerisce di studiare, come esercizio, un **sistema dinamico lineare** a due dimensioni che descrive, al variare dei parametri, la **variazione temporale dell'amore o dell'odio tra due partner coinvolti in una relazione romantica**.

Definiamo x(t) come l'amore (o l'odio nel caso in cui sia negativo) di Romeo nei confronti di Giulietta al tempo "t" e y(t) l'amore (o l'odio) di Giulietta nei confronti di Romeo. Così abbiamo le seguenti **due equazioni differenziali del primo ordine:**

Romeo $\dot{x} = ax + by$

Giulietta $\dot{y} = cx + dy$

I **parametri "a" e "b" stabiliscono il comportamento di Romeo mentre "c" e "d" quello di Giulietta**; più precisamente "a" descrive l'attrazione (o repulsione) di Romeo causata dai suoi stessi sentimenti, mentre "b" l'attrazione causata dai sentimenti di Giulietta. **Romeo** può mostrare 4 comportamenti diversi in base al segno dei parametri "a" e "b":

Appassionato: $a>0$; $b>0$ (Romeo è spinto dai suoi stessi sentimenti così come da quelli di Giulietta)

Narcisistico: $a>0$; $b<0$ (Romeo è spinto ancora dai suoi sentimenti ma indietreggia a causa dei sentimenti di Giulietta)

Amanti prudenti: $a<0$; $b>0$ (Romeo si tira indietro sui suoi stessi sentimenti ma è incoraggiato da Giulietta)

Eremita: $a<0$; $b<0$ (Romeo si tira indietro sui suoi stessi sentimenti così come da Giulietta)

Esercizio:

Esplorare il modello sia analiticamente che con l'aiuto di NetLogo in corrispondenza di diversi valori dei parametri e cercare di capire **quali limitazioni impone la linearità delle equazioni**.

Nota: Flussi Non Lineari vs Flussi Lineari

Possono esserci **più punti fissi**

Sistema Non Lineare

Rabbits $\dot{x} = x(3 - x - 2y)$

Sheep $\dot{y} = y(2 - x - y)$

Solutions:

$x = 0, \quad y = 2$

$x = 1, \quad y = 1$

$x = 3, \quad y = 0$

$y = 0, \quad x = 0$

Lo Jacobiano varia per ogni punto fisso e consente di studiare il comportamento della traiettoria solo in **prossimità del punto fisso** (in quanto deriva da espansioni in serie di Taylor).

$$J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}.$$

Sistema Lineare

Romeo $\dot{x} = ax + by$

Giulietta $\dot{y} = cx + dy$

$$\begin{cases} ax + by = 0 \\ cx + dy = 0 \end{cases}$$

Quando il determinante della matrice dei coefficienti è diverso da zero, il sistema ha esattamente un **unico punto fisso**, che può essere trovato risolvendo il sistema di equazioni lineari. In questo caso il punto fisso si trova **all'origine** (0,0) poichè non ci sono termini costanti nelle equazioni (sistema omogeneo).

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Lo Jacobiano coincide con la matrice dei coefficienti e quindi descrive il comportamento del sistema anche a **distanze maggiori** dal punto fisso (non richiede alcuna espansione in serie di Taylor).

Nei sistemi dinamici lineari non possono emergere cicli limite. I cicli limite sono una caratteristica dei sistemi dinamici non lineari. Un sistema lineare può avere un comportamento periodico (come nel caso di un oscillatore armonico ideale senza attrito), ma questo comportamento non rappresenta un ciclo limite. **La sensibilità alle condizioni iniziali di un sistema lineare è generalmente bassa**, nel senso che piccole variazioni nelle condizioni iniziali portano a differenze proporzionalmente piccole nel comportamento a lungo termine del sistema. Dunque un sistema lineare, a qualunque numero di dimensioni, **non potrà mostrare comportamenti caotici e neanche biforcazioni** (vedi più avanti...)

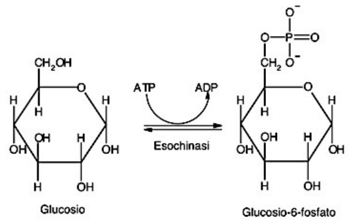
Ex.2 LA GLICOLISI

In the fundamental biochemical process called ***glycolysis***, living cells obtain energy by breaking down sugar. In intact yeast cells as well as in yeast or muscle extracts, glycolysis can proceed in an *oscillatory* fashion, with the concentrations of various intermediates waxing and waning with a period of several minutes. For reviews, see Chance et al. (1973) or Goldbeter (1980).

A simple model of these oscillations has been proposed by Sel'kov (1968). In dimensionless form, the equations are

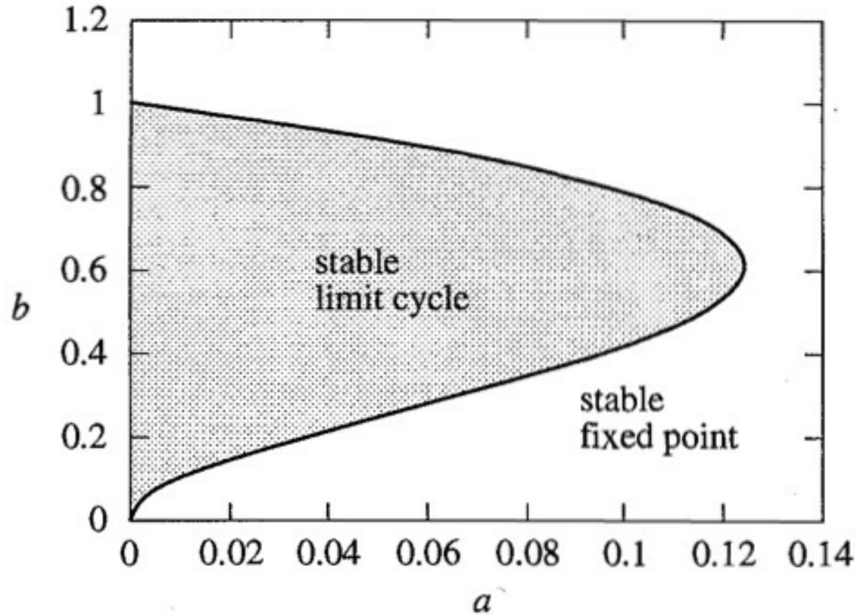
$$\dot{x} = -x + ay + x^2 y$$
$$\dot{y} = b - ay - x^2 y$$

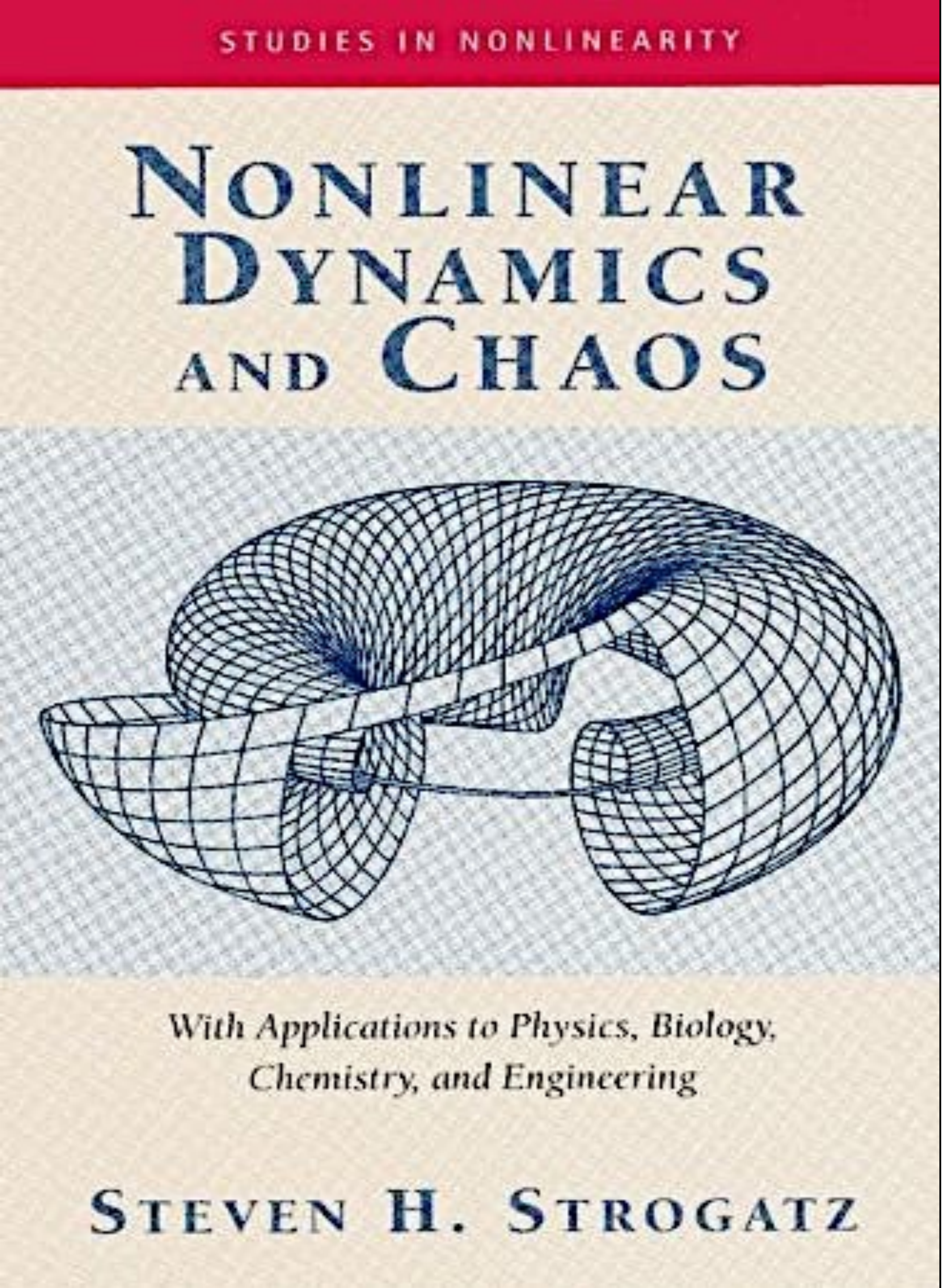
Valori tipici: a=0.08, b=0.6



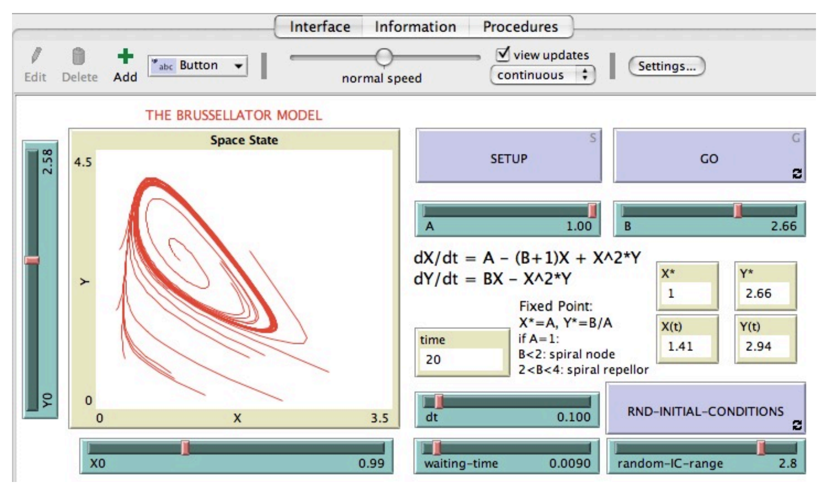
where x and y are the concentrations of ADP (adenosine diphosphate) and F6P (fructose-6-phosphate), and $a, b > 0$ are kinetic parameters.

Esercizio:
Esplorare il modello sia analiticamente che con l'aiuto di NetLogo



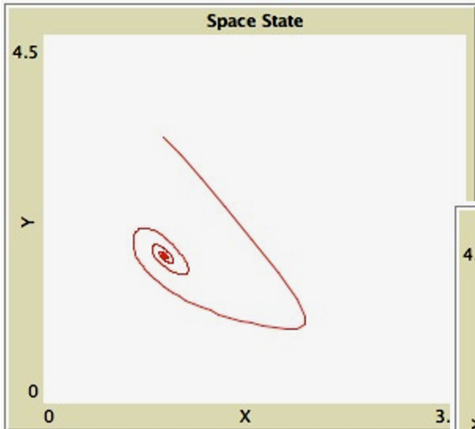


Sullo Strogatz potete trovare molti altri spunti per lo studio analitico e numerico di sistemi dinamici a 2 dimensioni…



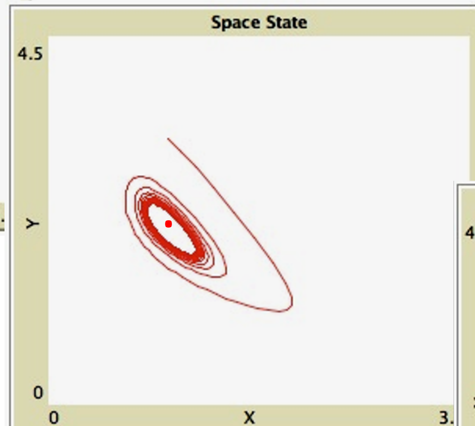
brussellator.nlogo

A=1, B=1.80



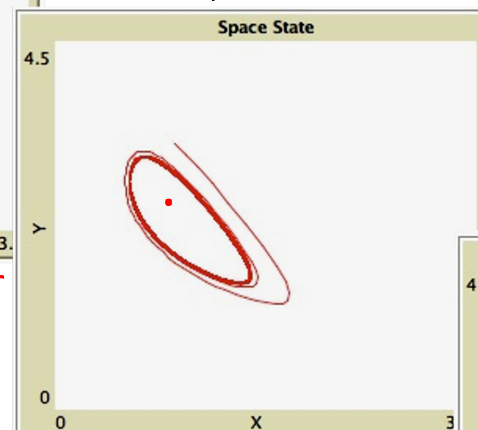
1 stable spiral node

A=1, B=2.15



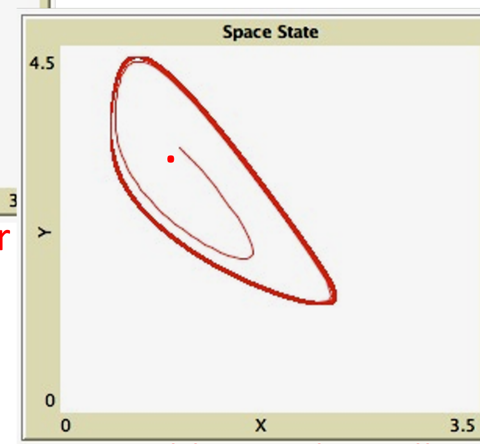
1 unstable spiral repellor
+
1 stable limit cycle

A=1, B=2.33



1 unstable spiral repellor
+
1 stable limit cycle

A=1, B=2.85



1 unstable spiral repellor
+
1 stable limit cycle

$$\dot{X} = A - (B+1)X + X^2 Y$$
$$\dot{Y} = BX - X^2 Y$$

1 punto fisso:

$$X_0 = A,\ Y_0 = B/A.$$

A=1,B<2 : stable spiral node

A=1, B>2 :
unstable spiral repellor + 1 stable limit cycle

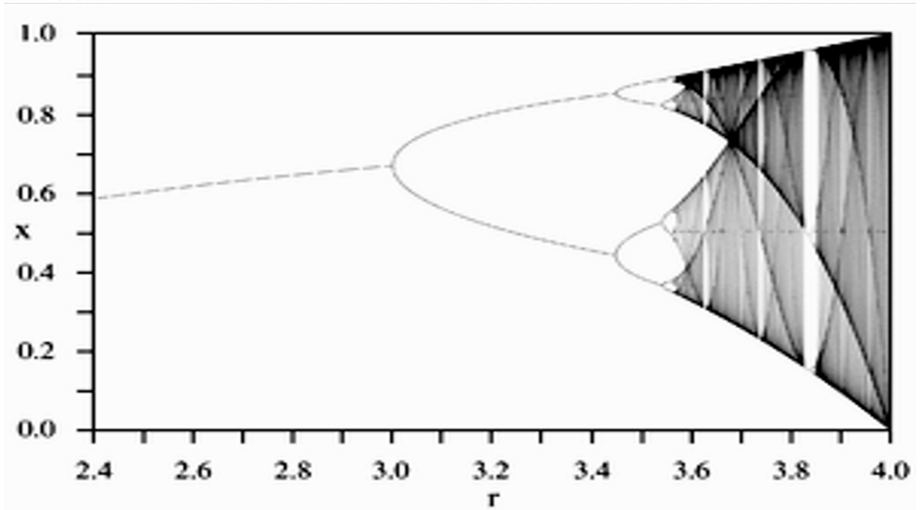
A=1, B=2 : Nasce il ciclo limite!

BIFORCAZIONE

Biforcazioni

3.17 Bifurcation Theory (vale per Flussi e Mappe)

We have seen that the characteristic values associated with a fixed point depend on the various parameters used to describe the system. As the parameters change, for example as we adjust a voltage in a circuit or the concentration of chemicals in a reactor, the nature of the characteristic values and hence the character of the fixed point may change. For example, an attracting node may become a repellor or a saddle point. The study of how the character of fixed points (and other types of state space attractors) change as parameters of the system change is called ***bifurcation theory***. (Recall that the term *bifurcation* is used to describe any sudden change in the dynamics of the system. When a fixed point changes character as parameter values change, the behavior of trajectories in the neighborhood of that fixed point will change. Hence the term bifurcation is appropriate here.) Being able to classify and understand the various possible bifurcations is an important part of the study of nonlinear dynamics. However, the theory, as it is presently developed, is rather limited in its ability to predict the kinds of bifurcations that will occur and the parameter values at which the bifurcations take place for a particular system. Description, however, is the first step toward comprehension and understanding.



We should also emphasize that simple bifurcation theory treats only the changes in stability of a particular attractor (or, as we shall see in Chapter 4, a particular basin of attraction). Since in general a system may have, for fixed parameter values, several attractors in different parts of state space, we often need to consider the overall dynamical system (that is, its "global" properties) to see what happens to trajectories when a bifurcation occurs.

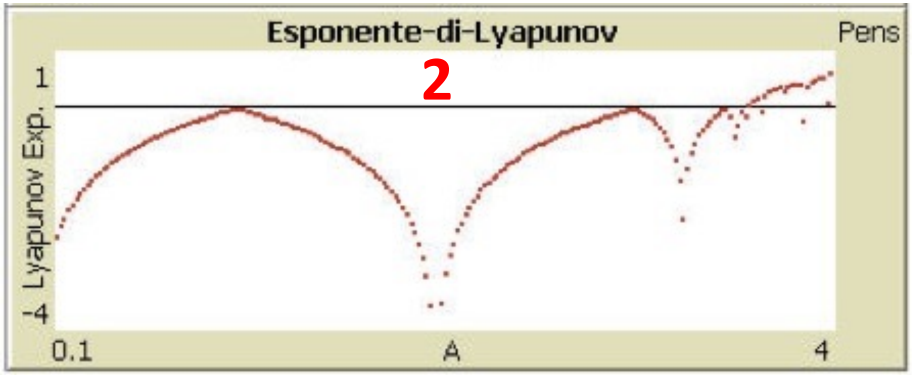
To keep track of what is happening as the control parameter is varied, we will use two types of diagrams. One type, which we have seen before, is the bifurcation
1 diagram, in which we plot the location of the fixed point (or points) as a function of the control parameter. In the second type of diagram, we plot the characteristic
2 values of the fixed point as a function of the control parameter.

To see how this kind of analysis proceeds, let us begin with the one-dimensional state space case. In a one-dimensional state space, a fixed point has just one characteristic value λ. The crucial assumption in the analysis is that λ varies smoothly (continuously) as some parameter, call it μ, varies. For example, if $\lambda(\mu) < 0$ for some value of μ, then the fixed point is a node. As μ changes, λ might increase (become less negative), going through zero, and then become positive. The node then changes to a repellor when $\lambda > 0$.

Ex. Flussi Dissip. 1D

Es.Mappa Logistica

$$x_{n+1} = Ax_n(1 - x_n)$$



Biforcazioni in 1D

Let us consider a specific example:

control parameter

Flusso dissipativo a una dimensione $\dot{x} = \mu - x^2$ (3.17-3)

For μ positive, there are two fixed points: one at $x = +\sqrt{\mu}$, the other at $x = -\sqrt{\mu}$. For μ negative there are no fixed points (assuming, of course, that x is a real number). If we use Eq. (3.6-3), which defines the characteristic value for a fixed point, to find the characteristic value of the two fixed points (for $\mu > 0$), we see that the fixed point at $x = -\sqrt{\mu}$ is a repellor, while the fixed point at $x = +\sqrt{\mu}$ is a node.

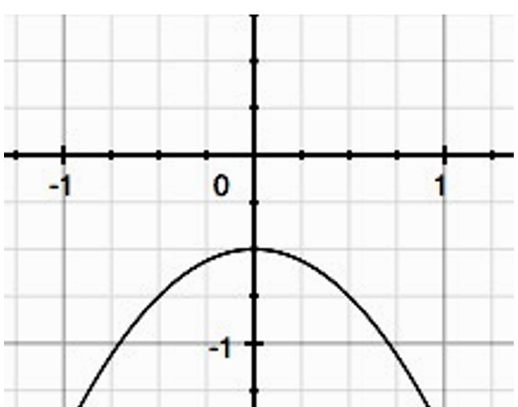
$$\lambda = \left.\frac{df(X)}{dX}\right|_{X=X_0}$$

$$\frac{df(X)}{dX} = -2x$$

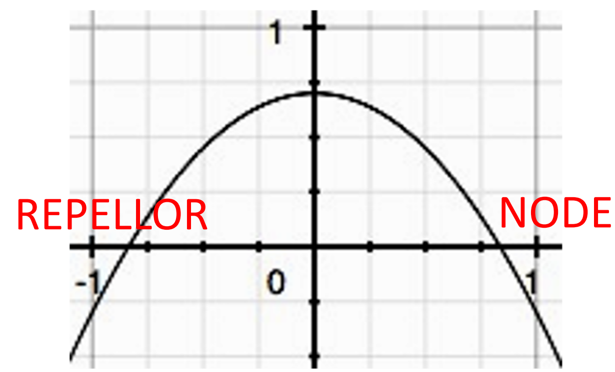
$$\lambda(+\sqrt{\mu}) < 0$$

$$\lambda(-\sqrt{\mu}) > 0$$

$\mu < 0$



$\mu > 0$



Biforcazioni in 1D

Let us consider a specific example:

control parameter

Flusso a una dimensione $$\dot{x} = \mu - x^2 \qquad (3.17\text{-}3)$$

For μ positive, there are two fixed points: one at $x = +\sqrt{\mu}$, the other at $x = -\sqrt{\mu}$. For μ negative there are no fixed points (assuming, of course, that x is a real number). If we use Eq. (3.6-3), which defines the characteristic value for a fixed point, to find the characteristic value of the two fixed points (for $\mu > 0$), we see that the fixed point at $x = -\sqrt{\mu}$ is a repellor, while the fixed point at $x = +\sqrt{\mu}$ is a node.

If we start with $\mu < 0$ and let it increase, we find that a bifurcation takes place at $\mu = 0$. At that value of the parameter we have a saddle point, which then changes into a repellor-node pair as μ becomes positive. We say that we have a ***repellor-node bifurcation*** at $\mu = 0$.

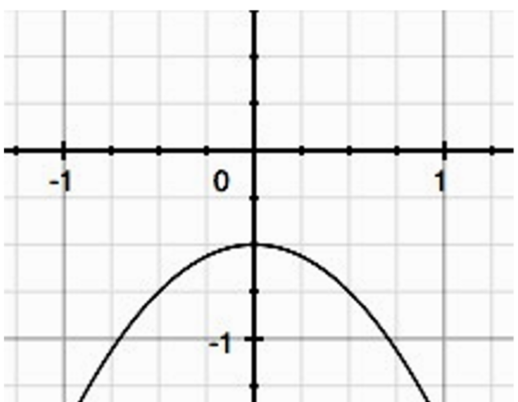
$$\lambda = \left.\frac{df(X)}{dX}\right|_{X=X_o}$$

$$\frac{df(X)}{dX} = -2x$$

$$\lambda(+\sqrt{\mu}) < 0$$

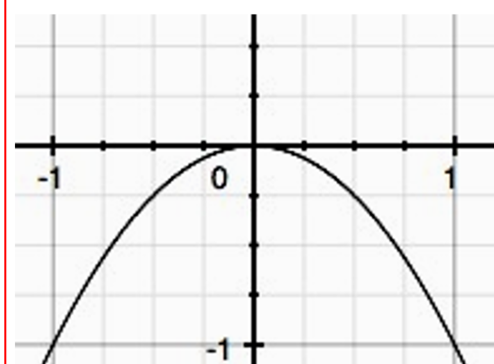
$$\lambda(-\sqrt{\mu}) > 0$$

$\mu < 0$

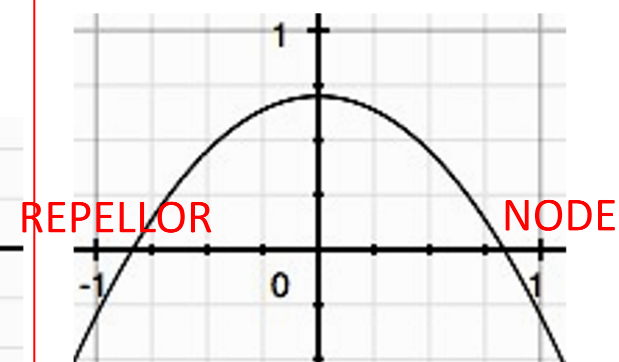


$\mu = 0$

BIFORCAZIONE!



$\mu > 0$



Biforcazioni in 1D

Let us consider a specific example:

control parameter

Flusso a una dimensione $$\dot{x} = \mu - x^2 \qquad (3.17\text{-}3)$$

For μ positive, there are two fixed points: one at $x = +\sqrt{\mu}$, the other at $x = -\sqrt{\mu}$. For μ negative there are no fixed points (assuming, of course, that x is a real number). If we use Eq. (3.6-3), which defines the characteristic value for a fixed point, to find the characteristic value of the two fixed points (for $\mu > 0$), we see that the fixed point at $x = -\sqrt{\mu}$ is a repellor, while the fixed point at $x = +\sqrt{\mu}$ is a node.

$$\lambda = \left.\frac{df(X)}{dX}\right|_{X=X_o}$$

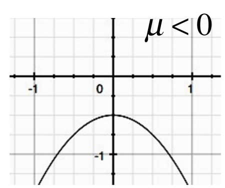
$$\frac{df(X)}{dX} = -2x$$

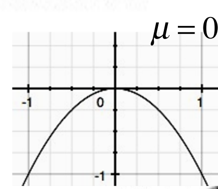
$$\lambda(+\sqrt{\mu}) < 0$$

$$\lambda(-\sqrt{\mu}) > 0$$

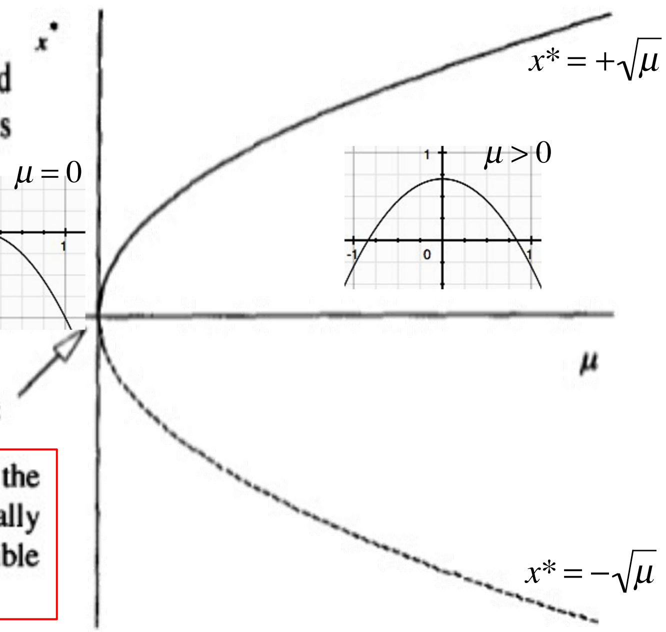
If we start with $\mu < 0$ and let it increase, we find that a bifurcation takes place at $\mu = 0$. At that value of the parameter we have a saddle point, which then changes into a repellor-node pair as μ becomes positive. We say that we have a ***repellor-node bifurcation*** at $\mu = 0$.

Fig. 3.14. The bifurcation diagram for the repellor-node (saddle-node) bifurcation. The solid line indicates the x value for the node as a function of the parameter value. The dashed line is for the repellor. Note that there is no fixed point at all for $\mu < 0$.





bifurcation point



Nota: Note that at the repellor-node bifurcation point, the fixed point of the system is structurally unstable in the sense discussed in Section 3.6. Structurally unstable points are important because their existence indicates a possible bifurcation.

In the nonlinear dynamics literature, the bifurcation just described is usually called a ***saddle-node bifurcation***, ***tangent bifurcation***, or a ***fold bifurcation***. The origin of these names will become apparent when we see analogous bifurcations in higher-dimensional state spaces. For example, if we imagine the curves in Fig. 3.14 as being the cross section of a piece of paper extending into and out of the plane of the page, then the bifurcation point represents a "fold" in the piece of paper. Also, Fig. 3.5 shows how the function in question becomes tangent to the *x* axis at the bifurcation point.

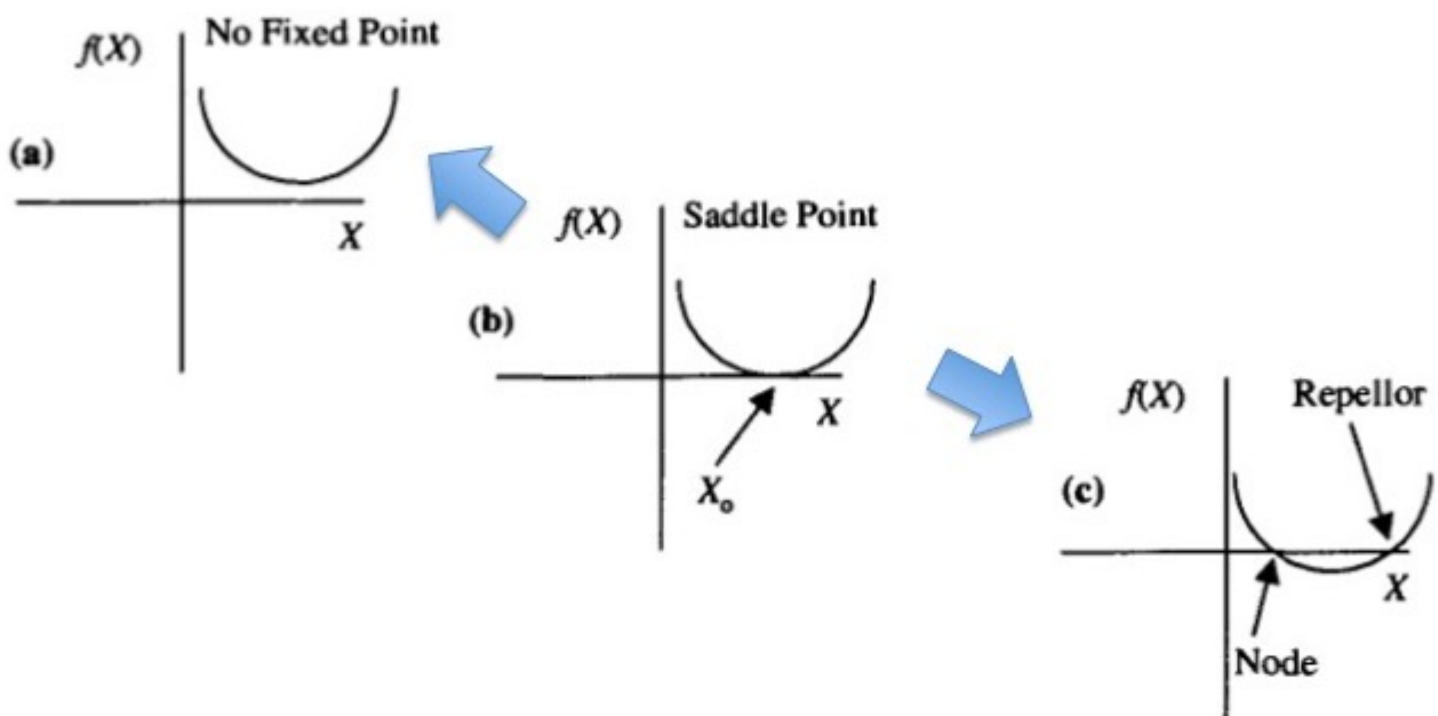
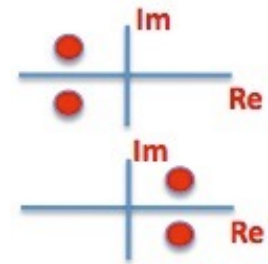


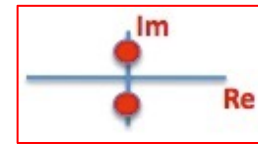
Fig. 3.5. In one-dimensional state spaces, a saddle point, the point X_o in (b), is structurally unstable. A small change in the function $f(X)$, for example pushing it up or down along the vertical axis, either removes the fixed point (a), or changes it into a node and a repellor (c).

Biforcazioni in 2D

Limit Cycle Bifurcations



As we saw earlier, a fixed point in a two-dimensional state space may also have complex-valued characteristic values for which the trajectories have spiral-type behavior. A bifurcation occurs when the characteristic values move from the left-hand side of the complex plane to the right-hand side; that is, the bifurcation occurs when the real part of the characteristic value goes to 0.



We can also have limit cycle behavior in two-dimensional systems. The birth and death of a limit cycle are bifurcation events. The birth of a stable limit cycle is called a ***Hopf bifurcation*** (named after the mathematician E. Hopf). (Although this type of bifurcation was known and understood by Poincaré and later studied by the Russian mathematician A. D. Andronov in the 1930s, Hopf was the first to extend these ideas to higher-dimensional state spaces.) Since we can use a Poincaré section to study a limit cycle and since for a two-dimensional state space, the Poincaré section is just a line segment, the bifurcations of limit cycles can be studied by the same methods used for studying bifurcations of one-dimensional dynamical systems.

Es: BRUSSELLATOR

A Hopf bifurcation can be modeled using the following normal form equations:

Flusso dissipativo a due dimensioni

$$\dot{x}_1 = -x_2 + x_1\{\mu - (x_1^2 + x_2^2)\} \tag{3.17-5a}$$

$$\dot{x}_2 = +x_1 + x_2\{\mu - (x_1^2 + x_2^2)\} \tag{3.17-5b}$$

$$\dot{x}_1 = -x_2 + x_1\{\mu - (x_1^2 + x_2^2)\} \quad (3.17\text{-}5a)$$

$$\dot{x}_2 = +x_1 + x_2\{\mu - (x_1^2 + x_2^2)\} \quad (3.17\text{-}5b)$$

Esiste chiaramente un punto fisso nell'origine...

The geometric form of the trajectories is clearer if we change from (x_1, x_2) coordinates to polar coordinates (r,θ) defined in the following equations and illustrated in Fig. 3.18.

$$r = \sqrt{(x_1^2 + x_2^2)}$$
$$\tan\theta = \frac{x_2}{x_1} \quad (3.17\text{-}6)$$

Distanza dal punto fisso nell'origine

Using these polar coordinates, we write Eqs. (3.17-5) as

$$\dot{r} = r\{\mu - r^2\} \equiv f(r) \quad (3.17\text{-}7a)$$

cubica

$$\dot{\theta} = 1 \rightarrow \theta(t) = \theta_o + t \quad (3.17\text{-}7b)$$

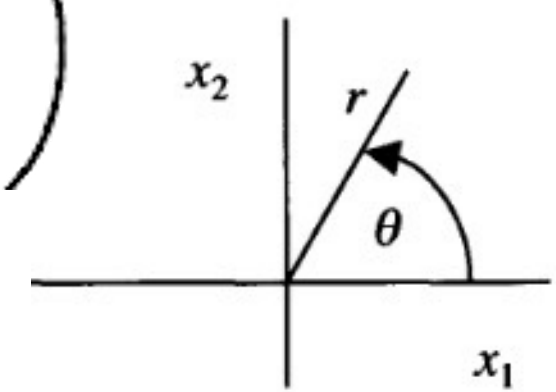


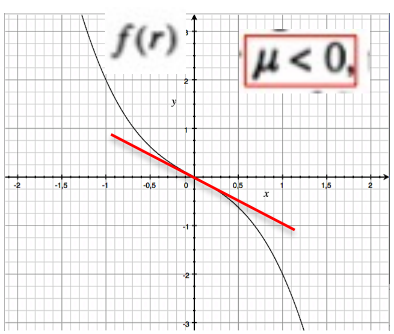
Fig. 3.18. The definition of polar coordinates. r is the length of the radius vector from the origin. θ is the angle between the radius vector and the positive x_1 axis.

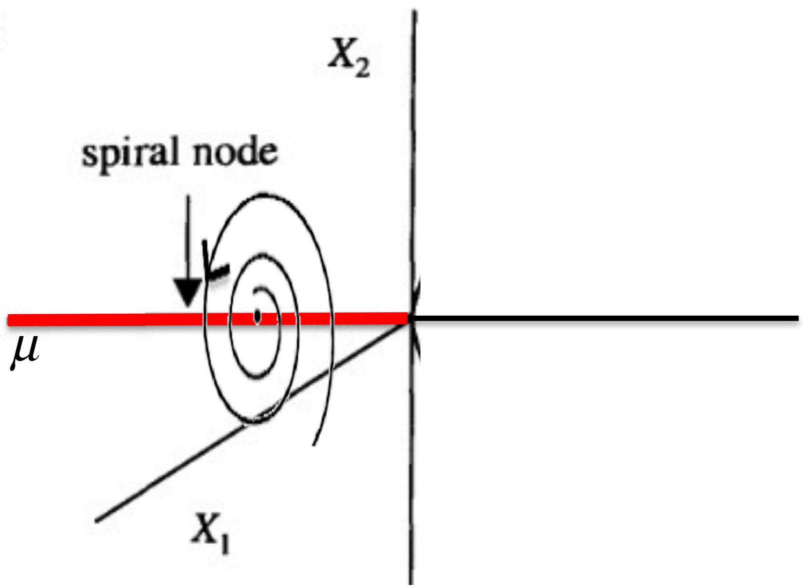
Now let us interpret the geometric nature of the trajectories that follow from Eqs. (3.17-7). The solution to Eq. (3.17-7b) is simply

$$\theta(t) = \theta_o + t \tag{3.17-8}$$

that is, the angle continues to increase with time as the trajectory spirals around the origin. For $\mu < 0$, there is just one fixed point for r, namely $r = 0$. By evaluating the derivative of $f(r)$ with respect to r at $r = 0$, we see that the characteristic value is equal to μ. Thus, for $\mu < 0$, that derivative is negative, and the fixed point is stable. In fact, it is a spiral node.

Fig. 3.19.





$$\begin{cases} \dot{r} = r\{\mu - r^2\} \equiv f(r) \\ \dot{\theta} = 1 \end{cases}$$

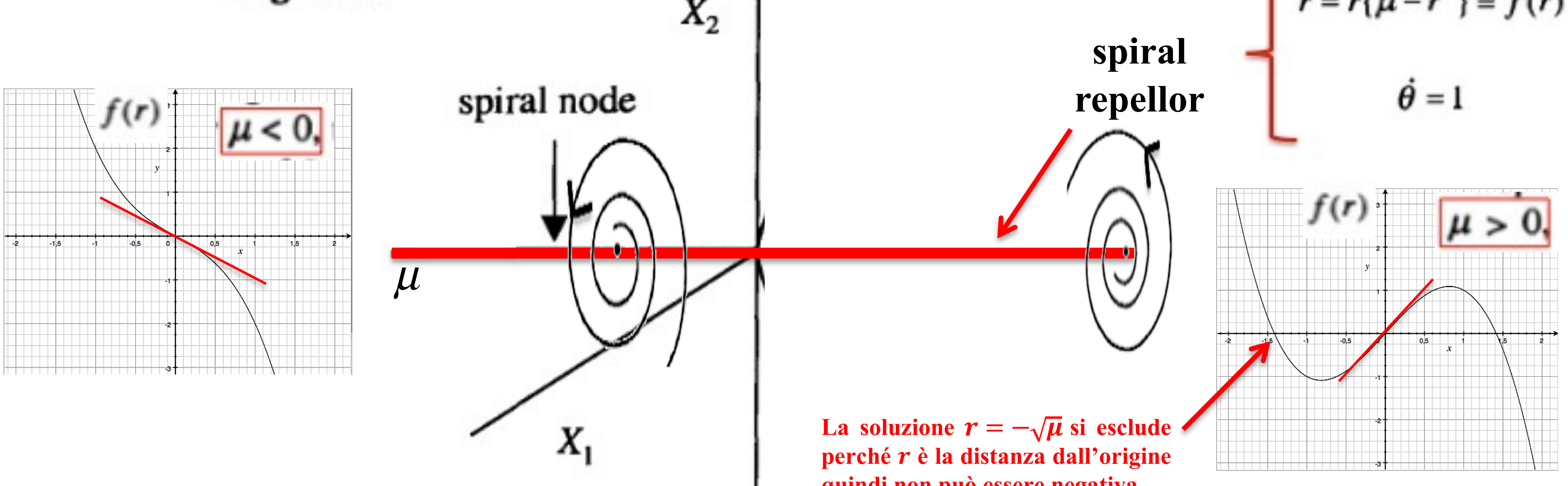
Now let us interpret the geometric nature of the trajectories that follow from Eqs. (3.17-7). The solution to Eq. (3.17-7b) is simply

$$\theta(t) = \theta_o + t \tag{3.17-8}$$

that is, the angle continues to increase with time as the trajectory spirals around the origin. For $\mu < 0$, there is just one fixed point for r, namely $r = 0$. By evaluating the derivative of $f(r)$ with respect to r at $r = 0$, we see that the characteristic value is equal to μ. Thus, for $\mu < 0$, that derivative is negative, and the fixed point is stable. In fact, it is a spiral node.

For $\mu > 0$, the fixed point at the origin is a spiral repellor; it is unstable; trajectories starting near the origin spiral away from it. There is, however, another fixed point for r, namely, $r = \sqrt{\mu}$.

Fig. 3.19.



Now let us interpret the geometric nature of the trajectories that follow from Eqs. (3.17-7). The solution to Eq. (3.17-7b) is simply

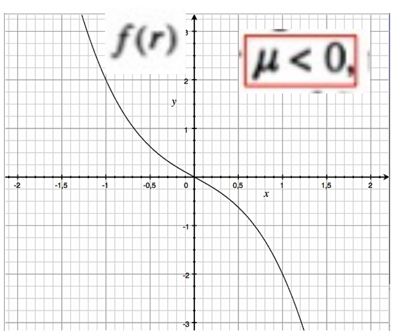
$$\theta(t) = \theta_o + t \qquad (3.17\text{-}8)$$

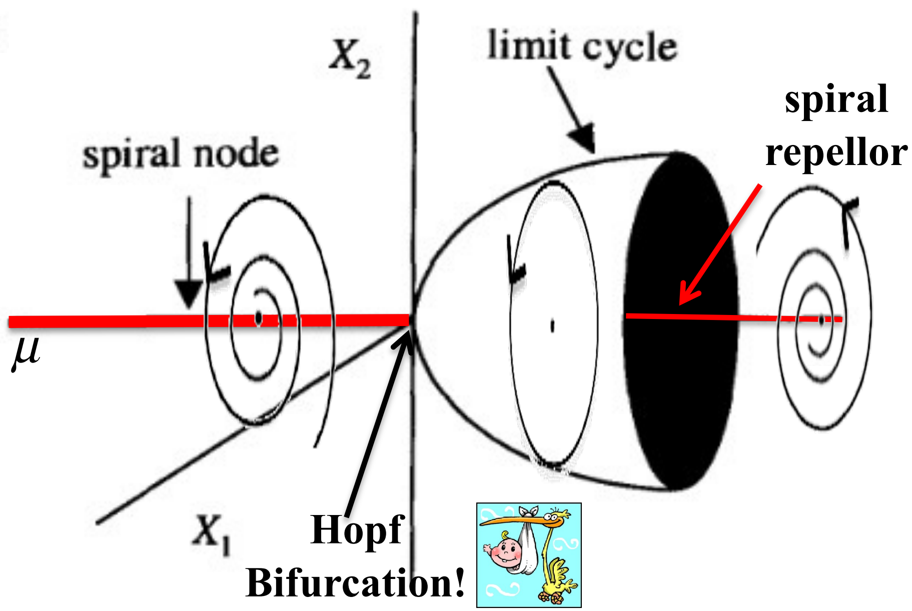
that is, the angle continues to increase with time as the trajectory spirals around the origin. For $\mu < 0$, there is just one fixed point for r, namely $r = 0$. By evaluating the derivative of $f(r)$ with respect to r at $r = 0$, we see that the characteristic value is equal to μ. Thus, for $\mu < 0$, that derivative is negative, and the fixed point is stable. In fact, it is a spiral node.

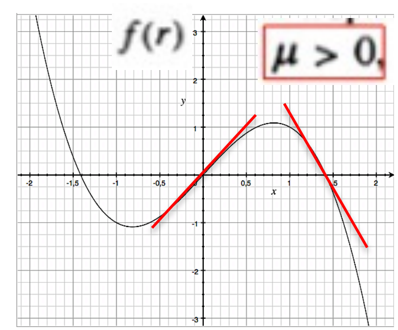
For $\mu > 0$, the fixed point at the origin is a spiral repellor; it is unstable; trajectories starting near the origin spiral away from it. There is, however, another fixed point for r, namely, $r = \sqrt{\mu}$. This fixed point for r corresponds to a limit cycle with a period of 2π [in the time units of Eqs. (3.17-7)]. We say that the limit cycle is born at the bifurcation value $\mu = 0$. Fig. 3.19 shows the bifurcation diagram for the Hopf bifurcation.

Fig. 3.19.

$$\begin{cases} \dot{r} = r\{\mu - r^2\} \equiv f(r) \\ \dot{\theta} = 1 \end{cases}$$



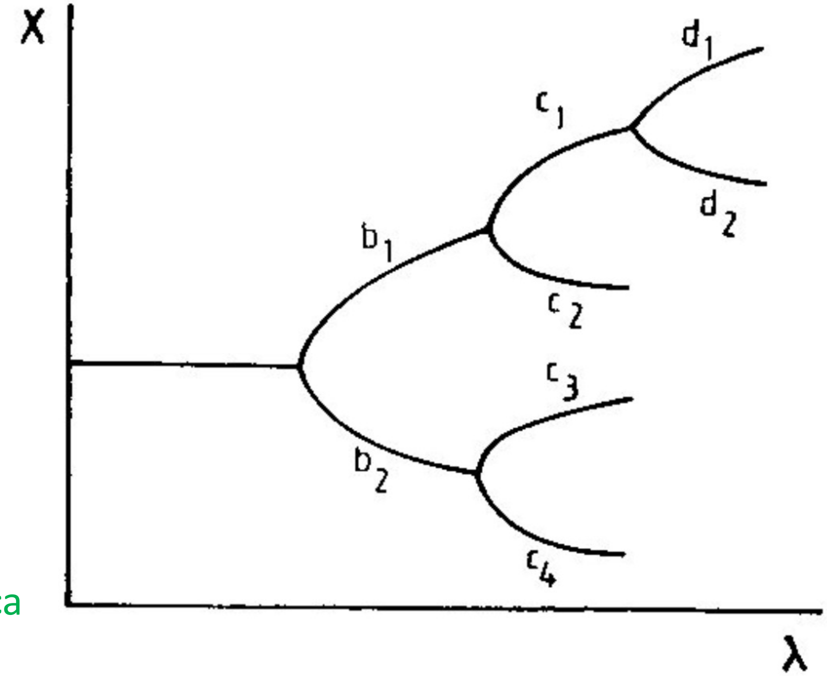


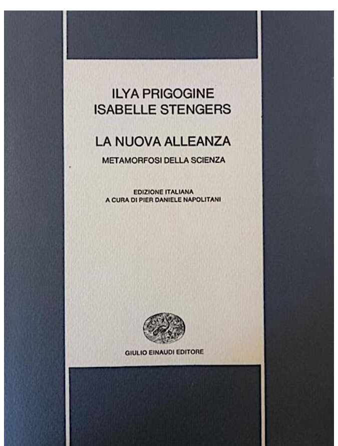


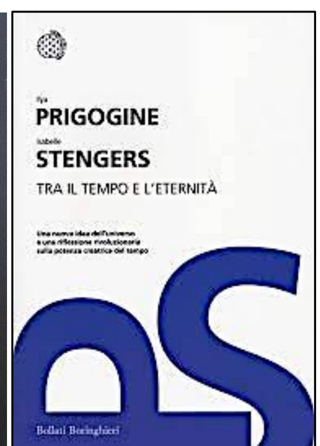
Biforcazioni e Strutture Dissipative

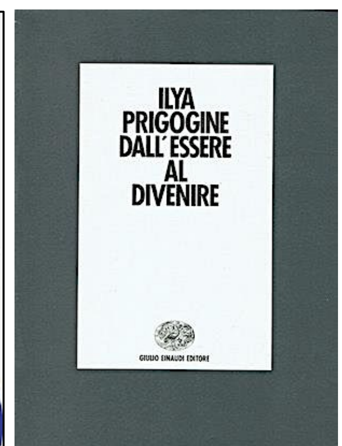
Sequenza di biforcazioni nei sistemi lontani dall'equilibrio

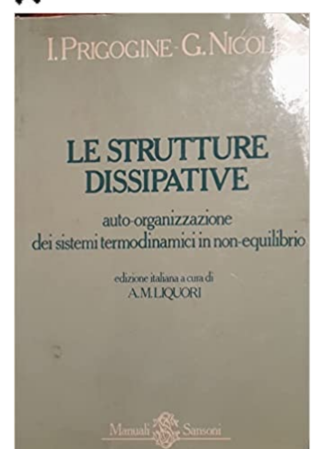
Ilya Prigogine
(1917-2003)
Nobel 1977 per la Chimica

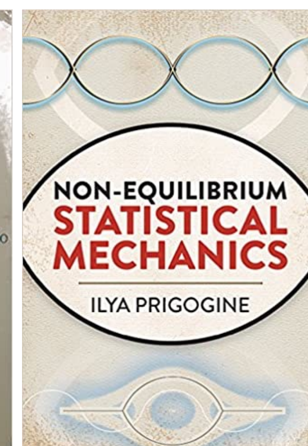












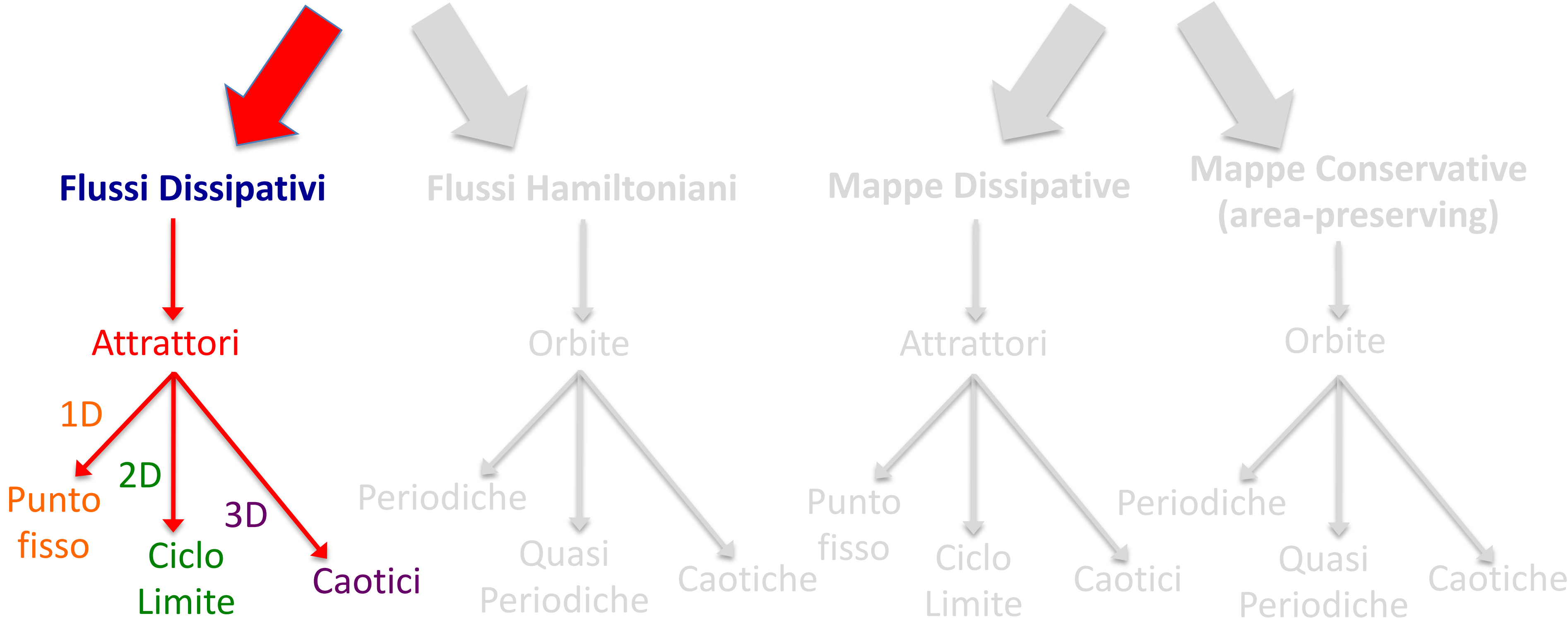
Classificazione dei Sistemi Dinamici

Sistemi dinamici continui (Flussi)

$$\dot{X} = f(X)$$

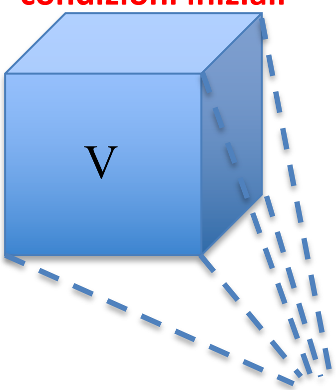
Sistemi dinamici discreti (Mappe)

$$x_{n+1} = f(x_n)$$



Flussi dissipativi in tre dimensioni

Cluster di condizioni iniziali



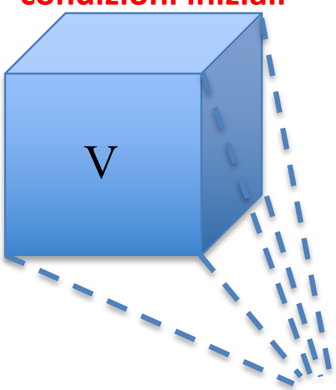
$$\frac{1}{V}\frac{dV}{dt} = \sum_{i=1}^{N}\frac{\partial f_i}{\partial x_i} \equiv div(f) \quad < 0$$

ATTRATTORI

- fixed points (dim.0)
- limit cycles (dim.1)
- quasiperiodic attractors (dim.2)
- chaotic attractors (dim…???)

Flussi dissipativi in tre dimensioni

Cluster di condizioni iniziali



$$\frac{1}{V}\frac{dV}{dt} = \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i} \equiv div(f) < 0$$

ATTRATTORI

- fixed points (dim.0)
- limit cycles (dim.1)
- quasiperiodic attractors (dim.2)
- chaotic attractors (fractal dimension between 2 and 3)

4

Three-Dimensional State Space and Chaos

4.1 Overview

In the previous chapter, we introduced some of the standard methods for analyzing dynamical systems described by systems of ordinary differential equations, but we limited the discussion to state spaces with one or two dimensions. We are now ready to take the important step to three dimensions. This is a crucial step, not because we live in a three-dimensional world (remember that we are talking about state space, not physical space), but because in three dimensions dynamical systems can behave in ways that are not possible in one or two dimensions. Foremost among these new possibilities is chaos.

First we will give a hand-waving argument (we could call it heuristic if we wanted to sound more sophisticated) that shows why chaotic behavior may occur in three dimensions. We will then discuss, in parallel with the treatment of the previous chapter, a classification of the types of fixed points that occur in three dimensions. However, we gradually wean ourselves from the standard analytic techniques and begin to rely more and more on graphic and geometrical (topological) arguments. This change reflects the flavor of current developments in dynamical systems theory. In fact, the main goal of this chapter is to develop geometrical pictures of trajectories, attractors, and bifurcations in three-dimensional state spaces.

4.2 Heuristics

We will describe, in a rather loose way, why three (or more) state space dimensions are needed to have chaotic behavior. First, we should remind ourselves that we are dealing with dissipative systems whose trajectories eventually approach an attractor. For the moment we are concerned only with the trajectories that have settled into the attracting region of state space. When we write about the divergence of nearby trajectories, we are concerned with the behavior of trajectories within the attracting region of state space.

In a somewhat different context we will need to consider sensitive dependence on initial conditions. Initial conditions that are not, in general, part of an attractor can lead to very different long-term behaviors on different attractors. Those behaviors, determined by the nature of the attractor (or attractors), might be time-independent or periodic or chaotic.

As we saw in Chapter 1, chaotic behavior is characterized by the divergence of nearby trajectories in state space. As a function of time, the "separation" (suitably defined) between two nearby trajectories increases exponentially, at least for short times. The last restriction is necessary because we are concerned with systems whose trajectories stay within some bounded region of state space. The system does not "blow up." There are three requirements for chaotic behavior in such a situation:

1. no intersection of different trajectories;
2. bounded trajectories;
3. exponential divergence of nearby trajectories.

These conditions cannot be satisfied simultaneously in one- or two-dimensional state spaces. You should convince yourself that this is true by sketching some trajectories in a two-dimensional state space on a sheet of paper. However, in three dimensions, initially nearby trajectories can continue to diverge by wrapping over and under each other. Obviously sketching three-dimensional trajectories is more difficult. You might try using some relatively stiff wire to form some trajectories in three dimensions to show that all three requirements for chaotic behavior can be met. You should quickly discover that these requirements lead to trajectories that initially diverge, then curve back through the state space, forming in the process an intricate layered structure. Figure 4.1 is a sketch of diverging trajectories in a three-dimensional state space.

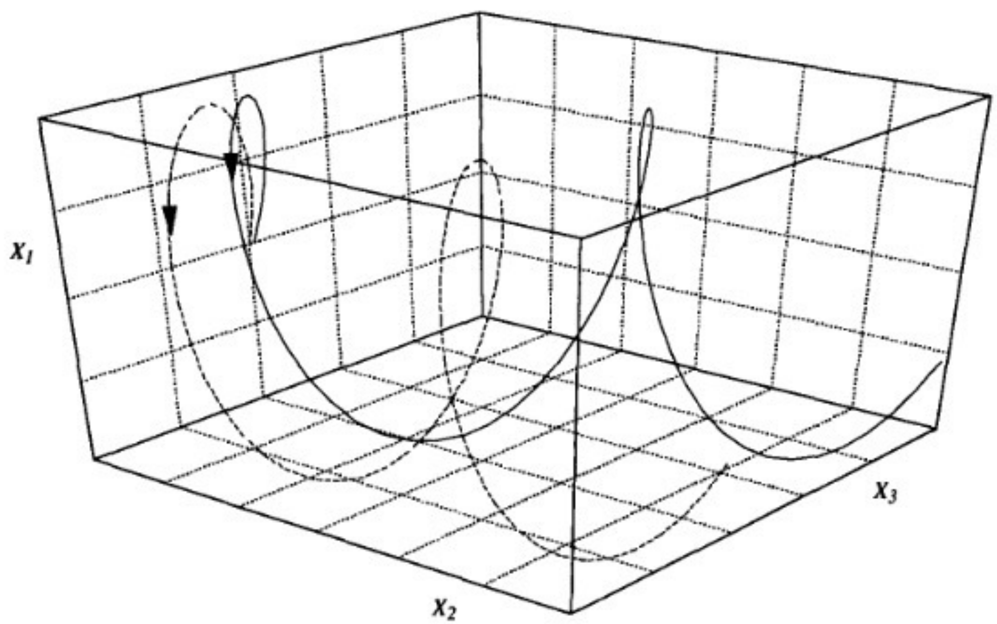


Fig. 4.1. A sketch of trajectories in a three–dimensional state space. Notice how two nearby trajectories can continue to behave quite differently from each other yet remain bounded by weaving in and out and over and under each other.

The notion of exponential divergence of nearby trajectories is made formal by introducing the ***Lyapunov exponent***. If two nearby trajectories on a chaotic attractor start off with a separation d_0 at time $t = 0$, then the trajectories diverge so that their separation at time t, denoted by $d(t)$, satisfies the expression

$$d(t) = d_0 e^{\lambda t} \tag{4.2-1}$$

The parameter λ in Eq. (4.2-1) is called the Lyapunov exponent for the trajectories. If λ is positive, then we say the behavior is chaotic. (Section 4.13 takes up the question of Lyapunov exponents in more detail.) From this definition of chaotic behavior, we see that chaos is a property of a collection of trajectories.

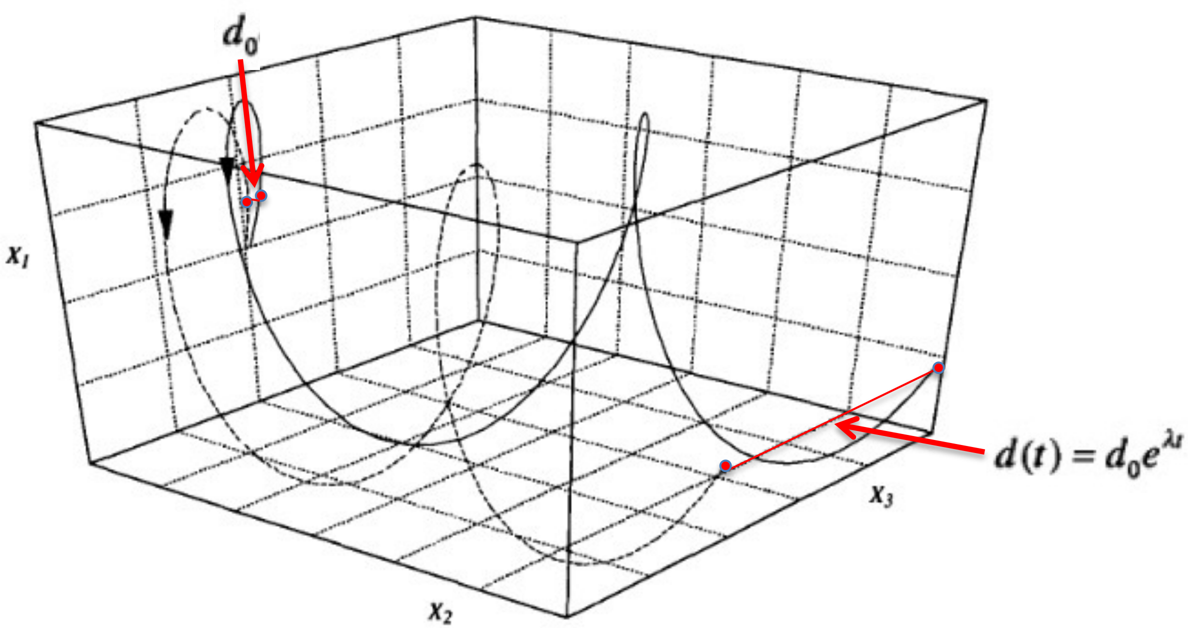
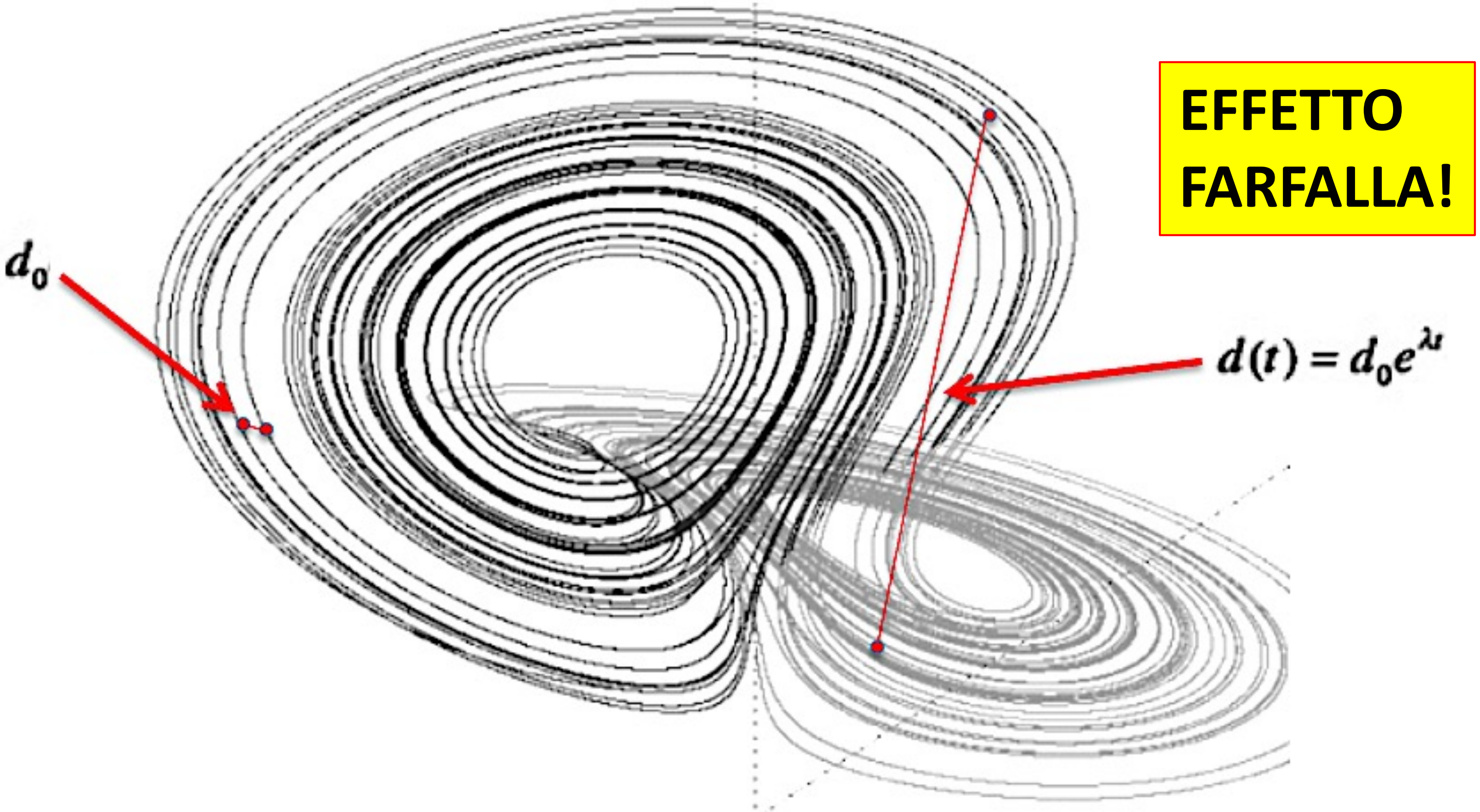
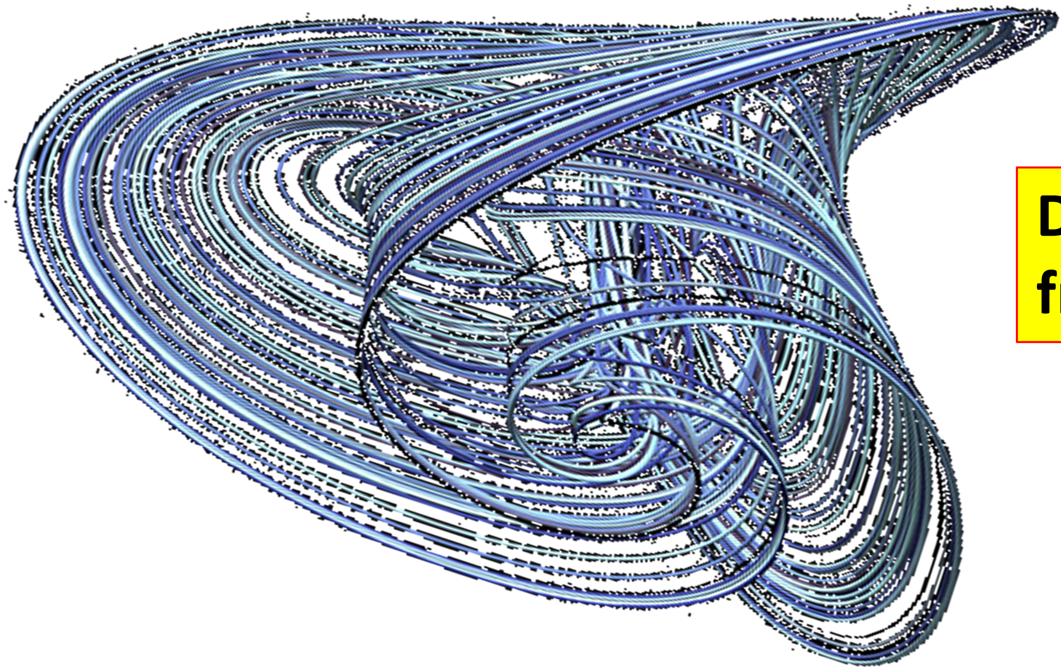


Fig. 4.1. A sketch of trajectories in a three-dimensional state space. Notice how two nearby trajectories can continue to behave quite differently from each other yet remain bounded by weaving in and out and over and under each other.

Chaos, however, also appears in the behavior of a single trajectory. As the trajectory wanders through the (chaotic) attractor in state space, it will eventually return near some point it previously visited. (Of course, it cannot return exactly to that point. If it did, then the trajectory would be periodic.) If the trajectories exhibit exponential divergence, then the trajectory on its second visit to a particular neighborhood will have subsequent behavior, quite different from its behavior on the first visit. Thus, the impression of the time record of this behavior will be one of nonreproducibility, nonperiodicity, in short, of chaos.



The crucial feature of state space with three or more dimensions that permits chaotic behavior is the ability of trajectories to remain within some bounded region by intertwining and wrapping around each other (without intersecting!) and without repeating themselves exactly. Clearly the geometry associated with such trajectories is going to be strange. In fact, such attractors are now called ***strange attractors***. In Chapter 9, we will give a more precise definition of a strange attractor in terms of the notion of fractal dimension. If the behavior on the attractor is chaotic, that is, if the trajectories on the attractor display exponential divergence of nearby trajectories (on the average), then we say the attractor is chaotic. Many authors use the terms *strange attractor* and ***chaotic attractor*** interchangeably, but in principle they are distinct [GOP84].



Dimensione frazionaria!

4.4 Three-Dimensional Dynamical Systems

We will now introduce some of the formalism for the description of a dynamical system with three state variables. We call a dynamical system three-dimensional if it has three independent dynamical variables, the values of which at a given instant of time uniquely specify the state of the system. We assume that we can write the time-evolution equations for the system in the form of the standard set of first-order ordinary differential equations. (Dynamical systems modeled by iterated map functions will be discussed in Chapter 5.) Here we will use x with a subscript 1, 2, or 3 to identify the variables. This formalism can then easily be generalized to any number of dimensions simply by increasing the numerical range of the subscripts. The differential equations take the form

$$\begin{aligned} \dot{X} &= p(Y - X) \\ \dot{Y} &= -XZ + rX - Y \\ \dot{Z} &= XY - bZ \end{aligned}$$

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, x_3) \\ \dot{x}_2 = f_2(x_1, x_2, x_3) \\ \dot{x}_3 = f_3(x_1, x_2, x_3) \end{cases} \qquad (4.4\text{-}1)$$

The Lorenz model equations of Chapter 1 are of this form. Note that the three functions f_1, f_2, and f_3 do not involve time explicitly; again, we say that the system is autonomous.

As an aside, we note that some authors like to use a symbolic "vector" form to write the system of equations:

$$\dot{\vec{x}} = \vec{f}(\vec{x}) \qquad (4.4\text{-}2)$$

Here $\vec{x}$ stands for the three symbols $x_1, x_2 x_3$, and $\vec{f}$ stands for the three functions on the right-hand side of Eqs. (4.4-1).

The differential equations describing two-dimensional systems subject to a time-dependent "force" (and hence nonautonomous) can also be written in the form of Eq. (4.4-1) by making use of the "trick" introduced in Chapter 3: Suppose that the two-dimensional system is described by equations of the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, t) \\ \dot{x}_2 &= f_2(x_1, x_2, t) \end{aligned} \tag{4.4-3}$$

The trick is to introduce a third variable, $x_3 = t$. The three "autonomous" equations then become

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, x_3) \\ \dot{x}_2 &= f_2(x_1, x_2, x_3) \\ \dot{x}_3 &= 1 \end{aligned} \tag{4.4-4}$$

which are of the same form as Eq. (4.4-1). As we shall see, this trick is particularly useful when the time-dependent term is periodic in time.

Exercise 4.4-1. The "forced" van der Pol equation is used to describe an electronic triode tube circuit subject to a periodic electrical signal. The equation for $q(t)$, the charge oscillating in the circuit, can be put in the form

$$\frac{d^2 q}{dt^2} + \gamma(q)\frac{dq}{dt} + q(t) = g \sin \omega t$$

Use the trick introduced earlier to write this equation in the standard form of Eq. (4.4-1).

4.5 Fixed Points in Three Dimensions (dim = 0)

The fixed points of the system of Eqs. (4.4-1) are found, of course, by setting the three time derivatives equal to 0. [Two-dimensional forced systems, even if written in the three-dimensional form (4.4-4), do not have any fixed points because, as the last of Eqs. (4.4-4) shows, we never have $\dot{x}_3 = t = 0$. Thus, we will need other techniques to deal with them.] The nature of each of the fixed points is determined by the three characteristic values of the Jacobian matrix of partial derivatives evaluated at the fixed point in question. The Jacobian matrix is

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{pmatrix} \tag{4.5-1}$$

In finding the characteristic values of this matrix, we will generally have a cubic equation, whose roots will be the three characteristic values labeled $\lambda_1, \lambda_2, \lambda_3$.

Some mathematical details: The standard theory of cubic equations tells us that a cubic equation of the form

$$\lambda^3 + p\lambda^2 + q\lambda + r = 0 \tag{4.5-2}$$

can be changed to the "standard" form

$$x^3 + ax + b = 0 \tag{4.5-3}$$

by the use of the substitutions

$$\begin{aligned} x &= \lambda + p/3 \\ a &= \frac{1}{3}(3q - p^2) \\ b &= \frac{1}{27}(2p^3 - 9qp + 27r) \end{aligned} \tag{4.5-4}$$

If we now introduce

$$\begin{aligned} s &= \left(\frac{b^2}{4} + \frac{a^3}{27}\right) \\ A &= (-b/2 + \sqrt{s})^{\frac{1}{3}} \\ B &= (-b/2 - \sqrt{s})^{\frac{1}{3}} \end{aligned} \tag{4.5-5}$$

the three roots of the x equation can be written as

$$
\begin{aligned}
\lambda_1 &= A + B \\
\lambda_2 &= -\left(\frac{A+B}{2}\right)+\left(\frac{A-B}{2}\right)\sqrt{-3} \\
\lambda_3 &= -\left(\frac{A+B}{2}\right)-\left(\frac{A-B}{2}\right)\sqrt{-3}
\end{aligned}
\qquad (4.5\text{-}6)
$$

from which the characteristic values for the matrix can be found by working back through the set of substitutions. Most readers will be greatly relieved to know that we will not make explicit use of these equations. But it is important to know the form of the solutions.

the three roots of the x equation can be written as

$$
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\end{aligned}
\qquad (4.5\text{-}6)
$$

from which the characteristic values for the matrix can be found by working back through the set of substitutions. Most readers will be greatly relieved to know that we will not make explicit use of these equations. But it is important to know the form of the solutions.

There are three cases to consider:

"standard" form

$$x^3 + ax + b = 0$$

$$s = \left(\frac{b^2}{4} + \frac{a^3}{27}\right)$$

1. The three characteristic values are real and unequal ($s < 0$).
2. The three characteristic values are real and at least two are equal ($s = 0$).
3. There is one real characteristic value and two complex conjugate values ($s > 0$).

Case 2 is just a borderline case and need not be treated separately.

Punti Fissi in uno Spazio degli Stati a Tre Dimensioni

The four basic types of fixed points for a three-dimensional state space are:

1. *Node*. All the characteristic values are real and negative. All trajectories in the neighborhood of the node are attracted toward the fixed point without looping around the fixed point.

1s. ***Spiral Node***. All the characteristic values have negative real parts but two of them have nonzero imaginary parts (and in fact form a complex conjugate pair). The trajectories spiral around the node on a "surface" as they approach the node.

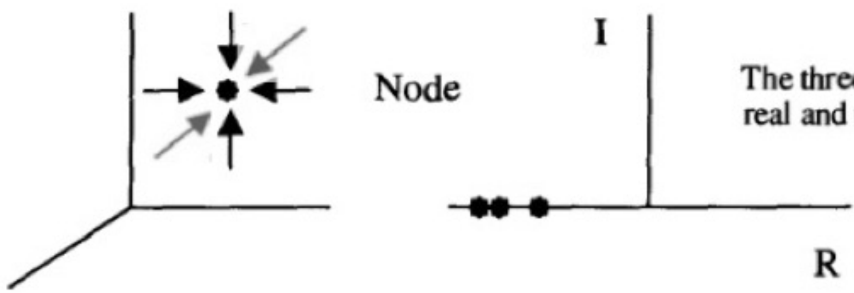
Equazione caratteristica:

$$\lambda^3 + p\lambda^2 + q\lambda + r = 0$$

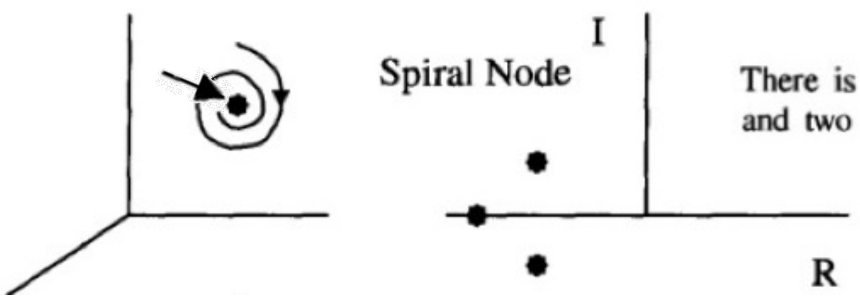
"standard" form

$$x^3 + ax + b = 0$$

$$s = \left(\frac{b^2}{4} + \frac{a^3}{27}\right)$$



The three characteristic values are real and unequal ($s < 0$).



There is one real characteristic value and two complex conjugate values ($s > 0$).

Punti Fissi in uno Spazio degli Stati a Tre Dimensioni

The four basic types of fixed points for a three-dimensional state space are:

2. ***Repellor***. All the characteristic values are real and positive. All trajectories in the neighborhood of the repellor diverge from the repellor.

2s. ***Spiral Repellor***. All the characteristic values have positive real parts, but two of them have nonzero imaginary parts (and in fact form a complex conjugate pair). Trajectories spiral around the repellor (on a "surface") as they are repelled from the fixed point.

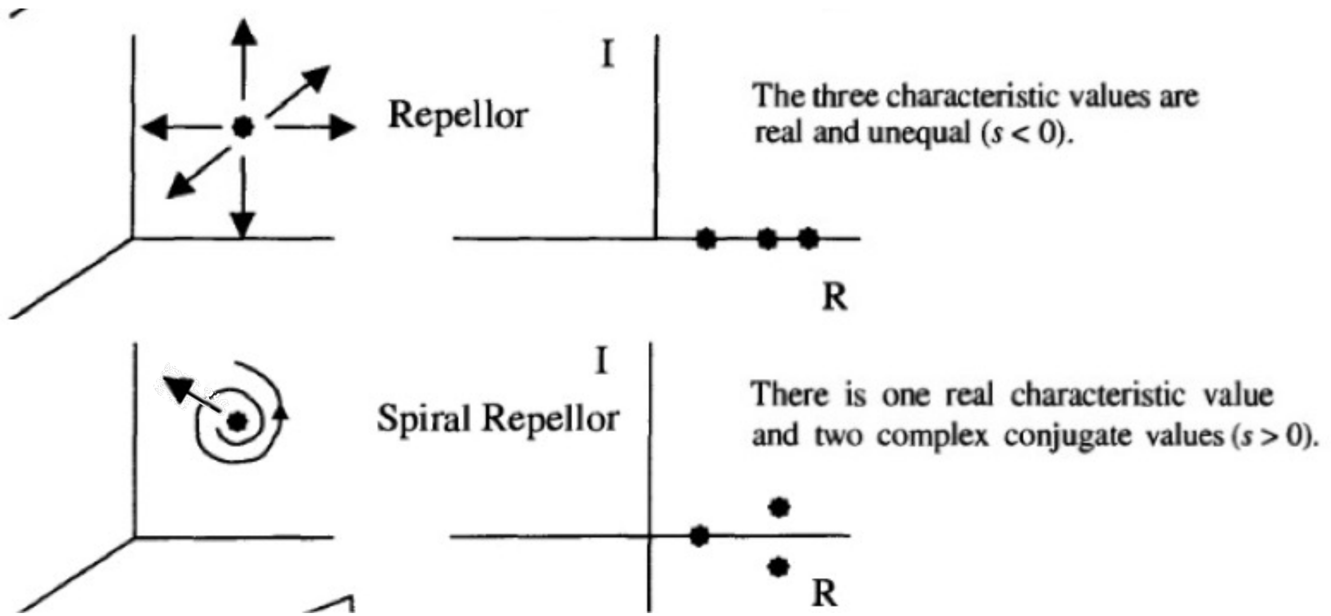
Equazione caratteristica:

$$\lambda^3 + p\lambda^2 + q\lambda + r = 0$$

"standard" form

$$x^3 + ax + b = 0$$

$$s = \left(\frac{b^2}{4} + \frac{a^3}{27}\right)$$



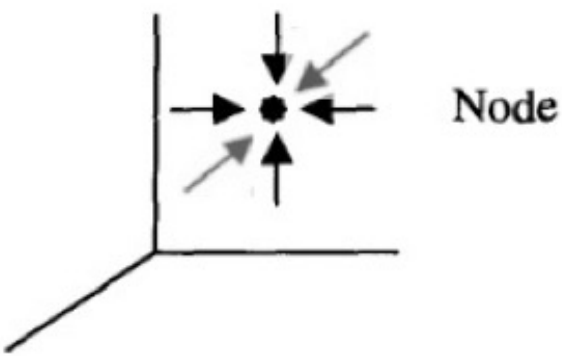
Punti Fissi in uno Spazio degli Stati a Tre Dimensioni

For state spaces with three or more dimensions, it is common to specify the so-called *index* of a fixed point.

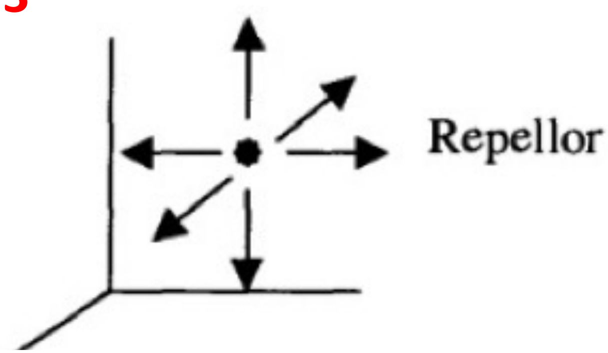
> The ***index*** of a fixed point is defined to be the number of characteristic values of that fixed point whose real parts are positive.

In more geometric terms, the index is equal to the spatial dimension of the out-set of that fixed point. For a node (which does not have an out-set), the index is equal to 0. For a repellor, the index is equal to 3 for a three-dimensional state space. A saddle point can have either an index of 1, if the out-set is a curve, or an index of 2, if the out-set is a surface as shown in Fig. 4.3.

Index = 0



Index = 3



Punti Fissi in uno Spazio degli Stati a Tre Dimensioni

3. ***Saddle point — index–1***. All characteristic values are real. One is positive and two are negative. Trajectories approach the saddle point on a surface (the in-set) and diverge along a curve (the out-set).

   3s. ***Spiral Saddle Point — index–1***. The two characteristic values with negative real parts form a complex conjugate pair. Trajectories spiral around the saddle point as they approach on the in-set surface.

4. ***Saddle point — index–2***. All characteristic values are real. Two are positive and one is negative. Trajectories approach the saddle point on a curve (the in-set) and diverge from the saddle point on a surface (the out-set).

   4s. ***Spiral Saddle Point — index–2***. The two characteristic values with positive real parts form a complex conjugate pair. Trajectories spiral around the saddle point on a surface (the out-set) as they diverge from the saddle point.