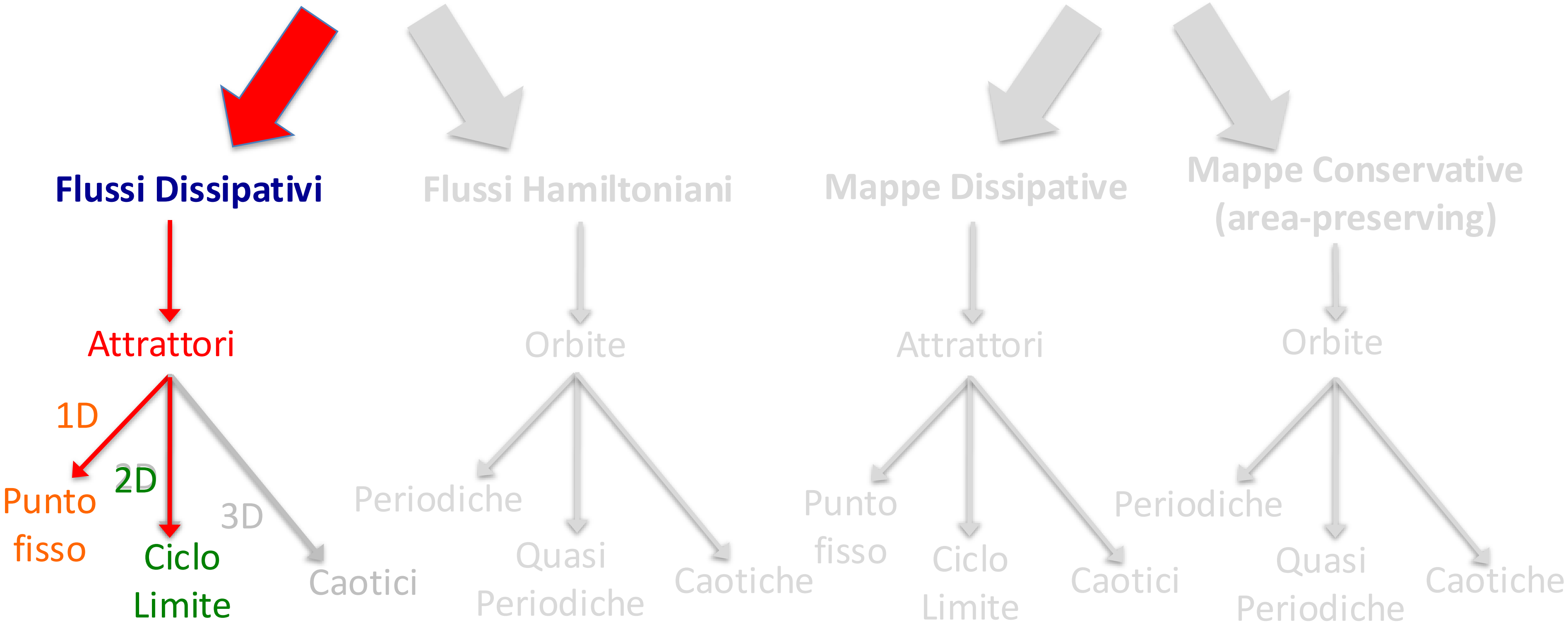
Classificazione dei Sistemi Dinamici

Sistemi dinamici continui (Flussi)

$$\dot{X} = f(X)$$

Sistemi dinamici discreti (Mappe)

$$x_{n+1} = f(x_n)$$



Flussi Dissipativi

- Attrattori
  - 1D Punto fisso
  - 2D Ciclo Limite
  - 3D Caotici

Flussi Hamiltoniani

- Orbite
  - Periodiche
  - Quasi Periodiche
  - Caotiche

Mappe Dissipative

- Attrattori
  - Punto fisso
  - Ciclo Limite
  - Caotici

Mappe Conservative (area-preserving)

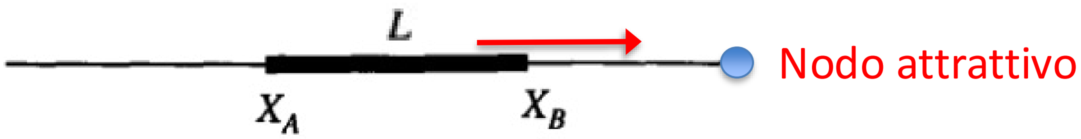
- Orbite
  - Periodiche
  - Quasi Periodiche
  - Caotiche

$$\boxed{\dot{X} = f(X)}$$

Flussi dissipativi in una dimensione

$$\frac{1}{L}\frac{dL}{dt} = \frac{1}{L}\left[f(X_B) - f(X_A)\right] = \frac{df(X)}{dX} < 0$$

fixed points (dim.0)

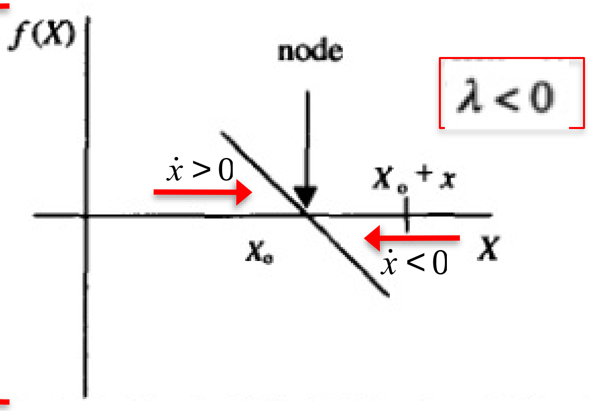

L
X_A
X_B
Nodo attrattivo

A "cluster of initial conditions," indicated by the heavy line, along the X axis.

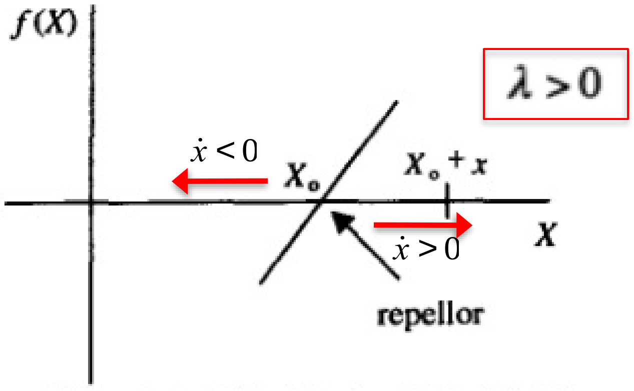
Riepilogo dei Punti Fissi in uno Spazio degli Stati a Una Dimensione

$$\dot{X}\Big|_{X=X_o} = f(X_o) = 0$$

Punti Fissi Strutturalmente Stabili



$f(X)$
node
$\lambda < 0$
$\dot{x} > 0$
$X_o + x$
X_o
$\dot{x} < 0$
X

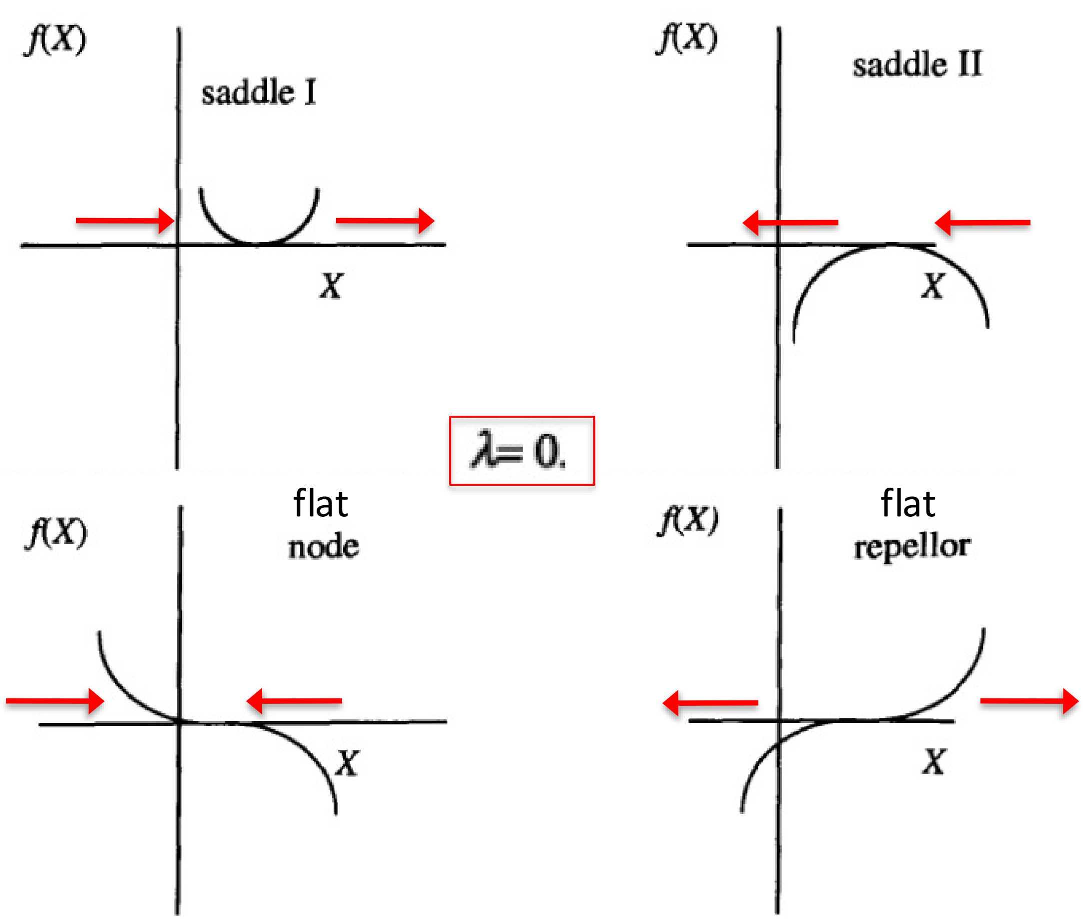


$f(X)$
$\lambda > 0$
$\dot{x} < 0$
X_o
$X_o + x$
$\dot{x} > 0$
X
repellor

$$\lambda = \frac{df(X)}{dX}\bigg|_{X=X_o}$$

Valore Caratteristico del Punto Fisso o Esponente di Lyapunov

Punti Fissi Strutturalmente Instabili

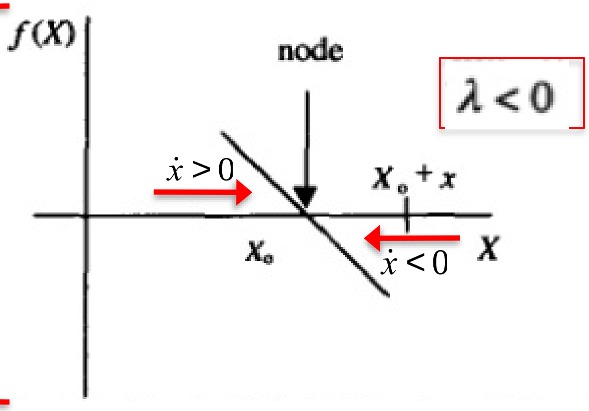


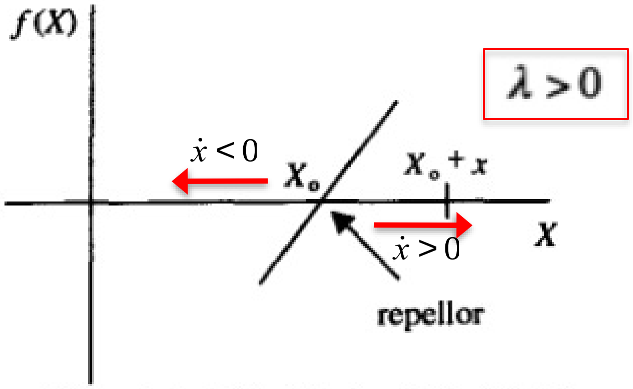
$f(X)$
saddle I
X
$f(X)$
saddle II
X
$\lambda = 0.$
$f(X)$
flat node
X
$f(X)$
flat repellor
X

Riepilogo dei Punti Fissi in uno Spazio degli Stati a Una Dimensione

$$\boxed{\dot{X}\big|_{X=X_o} = f(X_o) = 0}$$

Punti Fissi Strutturalmente Stabili


f(X)
node
λ<0
$\dot{x}>0$
X_o+x
X_o
$\dot{x}<0$
X

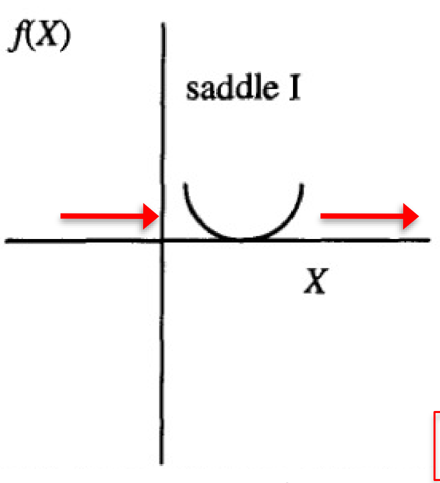

f(X)
λ>0
$\dot{x}<0$
X_o
X_o+x
$\dot{x}>0$
X
repellor

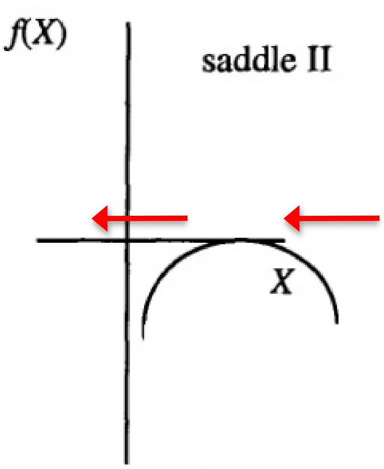
$$\dot{x} = \left.\frac{df}{dX}\right|_{X_o} x$$

Equazione linearizzata
per la distanza dal Punto Fisso

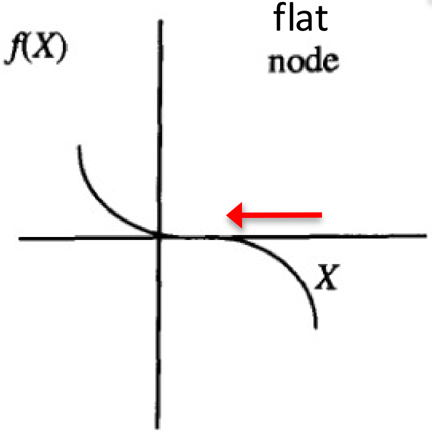
$$\boxed{x(t) = x(0)e^{\lambda t}}$$

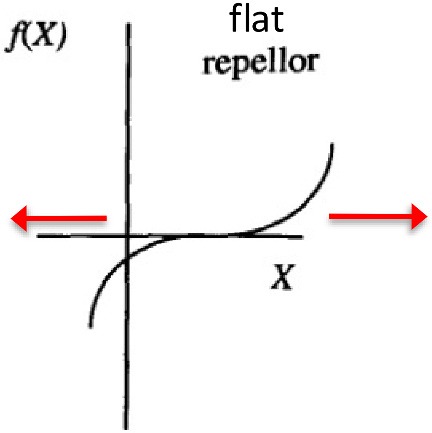
Punti Fissi Strutturalmente Instabili


f(X)
saddle I
X


f(X)
saddle II
X

$\boxed{\lambda = 0.}$


f(X)
flat
node
X


f(X)
flat
repellor
X

$$\dot X_1 = f_1(X_1.X_2)$$
$$\dot X_2 = f_2(X_1.X_2)$$

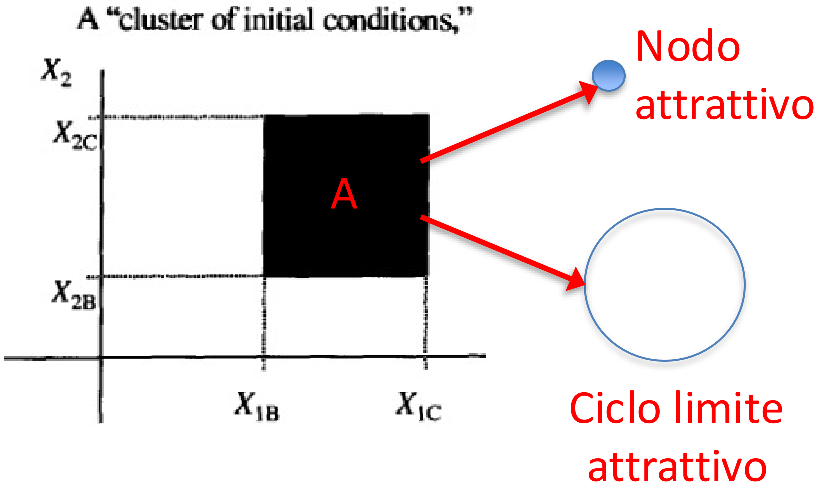
Flussi dissipativi in due dimensioni

Divergenza di $\vec{f}$

$$\frac{1}{A}\frac{dA}{dt} = \frac{\partial f_1}{\partial X_1} + \frac{\partial f_2}{\partial X_2} < 0$$

fixed points (dim.0)

limit cycles (dim.1)


A "cluster of initial conditions,"
X_2
X_{2C}
A
X_{2B}
X_{1B}
X_{1C}

Nodo attrattivo

Ciclo limite attrattivo

Studio della stabilità dei Punti Fissi in due dimensioni: il caso generale

$$\boxed{\begin{aligned}\dot X_1 &= f_1(X_1, X_2)\\ \dot X_2 &= f_2(X_1, X_2)\end{aligned}}$$

$$\begin{cases} f_1(X_{1o}, X_{2o}) = 0\\ f_2(X_{1o}, X_{2o}) = 0\end{cases} \longrightarrow$$

FIXED POINTS

PER CIASCUN PUNTO FISSO $(X_{1o}\, X_{2o})$:

$$\dot X_1 = f_1(X_1, X_2) = (X_1 - X_{1o})\frac{\partial f_1}{\partial X_1} + (X_2 - X_{2o})\frac{\partial f_1}{\partial X_2} + \dots \qquad (3.11\text{-}4a)$$

DISTANZA DELLA TRAIETTORIA DAL PUNTO FISSO LUNGO L'ASSE X_1

DISTANZA DELLA TRAIETTORIA DAL PUNTO FISSO LUNGO L'ASSE X_2

$$\dot X_2 = f_2(X_1, X_2) = (X_1 - X_{1o})\frac{\partial f_2}{\partial X_1} + (X_2 - X_{2o})\frac{\partial f_2}{\partial X_2} + \dots \qquad (3.11\text{-}4b)$$

$$\boxed{\begin{aligned}x_1 &= X_1 - X_{1o}\\ x_2 &= X_2 - X_{2o}\end{aligned}}$$

and ignoring all the higher-order derivative terms, we may write Eq. (3.11-4) as

Equazioni linearizzate attorno al punto fisso

$$\begin{cases}\dot x_1 = \dfrac{\partial f_1}{\partial x_1}x_1 + \dfrac{\partial f_1}{\partial x_2}x_2\\[2ex] \dot x_2 = \dfrac{\partial f_2}{\partial x_1}x_1 + \dfrac{\partial f_2}{\partial x_2}x_2\end{cases} \longrightarrow$$

soluzioni particolari

$$x_1(t) = Ce^{\lambda t}$$

Studio della stabilità dei Punti Fissi in due dimensioni: il caso generale

$$\boxed{\begin{aligned}\dot{X}_1 &= f_1(X_1, X_2)\\ \dot{X}_2 &= f_2(X_1, X_2)\end{aligned}}$$

$$\left\{\begin{aligned}f_1(X_{1o}, X_{2o}) &= 0\\ f_2(X_{1o}, X_{2o}) &= 0\end{aligned}\right. \longrightarrow$$

FIXED POINTS

PER CIASCUN PUNTO FISSO (X_{1o}, X_{2o}) :

Equazione Caratteristica

$$\lambda^2 - (f_{11} + f_{22})\lambda + (f_{11}f_{22} - f_{12}f_{21}) = 0$$

con $f_{ij} = \dfrac{\partial f_i}{\partial x_j}$

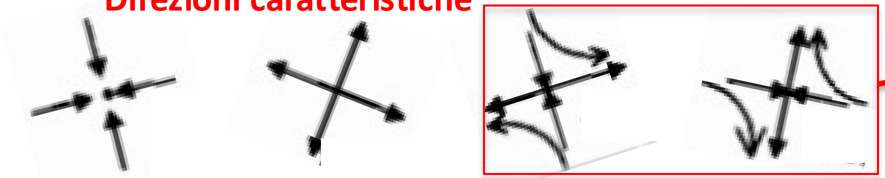
We call Eq. (3.11-11) the ***characteristic equation*** for λ, whose value depends only on the derivatives of the time evolution functions evaluated at the fixed point. Eq. (3.11-11) is a quadratic equation for λ and in general has two solutions, which we can write down from the standard quadratic formula:

2 Valori caratteristici

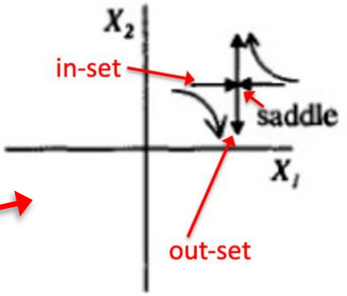
$$\boxed{\lambda_{\pm} = \frac{f_{11} + f_{22} \pm \sqrt{(f_{11} + f_{22})^2 - 4(f_{11}f_{22} - f_{12}f_{21})}}{2}}$$

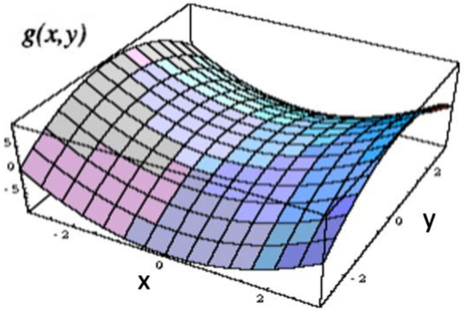
reali $\boxed{\lambda_{\pm} = R}$

Direzioni caratteristiche



saddle points





Studio della stabilità dei Punti Fissi in due dimensioni: il caso generale

$$\boxed{\begin{aligned}\dot{X}_1 &= f_1(X_1, X_2)\\ \dot{X}_2 &= f_2(X_1, X_2)\end{aligned}}$$

$$\begin{cases} f_1(X_{1o}, X_{2o}) = 0 \\ f_2(X_{1o}, X_{2o}) = 0 \end{cases} \longrightarrow$$

FIXED POINTS

PER CIASCUN PUNTO FISSO (X_{1o}, X_{2o}) :

Equazione Caratteristica

$$\lambda^2 - (f_{11} + f_{22})\lambda + (f_{11}f_{22} - f_{12}f_{21}) = 0 \qquad \text{con } f_{ij} = \frac{\partial f_i}{\partial x_j}$$

We call Eq. (3.11-11) the ***characteristic equation*** for λ, whose value depends only on the derivatives of the time evolution functions evaluated at the fixed point. Eq. (3.11-11) is a quadratic equation for λ and in general has two solutions, which we can write down from the standard quadratic formula:

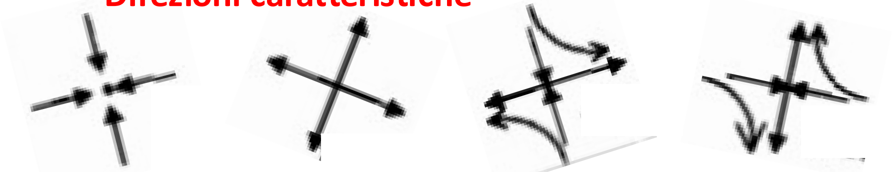
2 Valori caratteristici

$$\boxed{\lambda_{\pm} = \frac{f_{11} + f_{22} \pm \sqrt{(f_{11} + f_{22})^2 - 4(f_{11}f_{22} - f_{12}f_{21})}}{2}}$$

reali $\boxed{\lambda_{\pm} = R}$

$\boxed{\lambda_{\pm} = R \pm i\Omega}$ **complessi coniugati**

Direzioni caratteristiche



?

Studio della stabilità dei Punti Fissi in due dimensioni: il caso generale

$$\boxed{\begin{aligned}\dot{X}_1 &= f_1(X_1, X_2)\\ \dot{X}_2 &= f_2(X_1, X_2)\end{aligned}}$$

$$\left\{\begin{aligned} f_1(X_{1o}, X_{2o}) &= 0\\ f_2(X_{1o}, X_{2o}) &= 0\end{aligned}\right. \longrightarrow$$

FIXED POINTS

PER CIASCUN PUNTO FISSO $(X_{1o} X_{2o})$:

Equazione Caratteristica

$$\lambda^2 - (f_{11} + f_{22})\lambda + (f_{11} f_{22} - f_{12} f_{21}) = 0$$

con $f_{ij} = \dfrac{\partial f_i}{\partial x_j}$

Le distanze del punto rappresentativo del sistema dal punto fisso lungo i due assi dello spazio degli stati spiraleggiano nei dintorni del punto:

$$\boxed{\lambda_{\pm} = R \pm i\Omega}$$

$$\left\{\begin{aligned} x_1(t) &= F_1 e^{Rt} \sin Wt\\ x_2(t) &= F_2 e^{Rt} \sin Wt\end{aligned}\right.$$

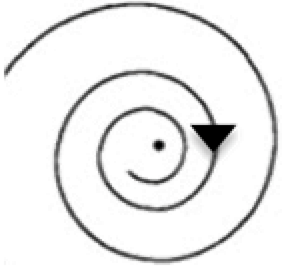
con:

$$F_1 \approx \frac{f_{12} x_2(0)}{\Omega}$$

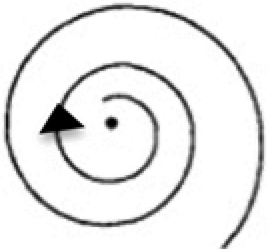
$$F_2 \approx \frac{f_{21} x_1(0)}{\Omega}$$

$$R = \frac{1}{2}(f_{11} + f_{22})$$

$$\Omega = \frac{1}{2}\sqrt{(f_{11} + f_{22})^2 - 4(f_{11} f_{22} - f_{12} f_{21})}$$



R<0: spiral node



R<0: spiral repellor

Studio della stabilità dei Punti Fissi in due dimensioni: il caso generale

$$\boxed{\begin{aligned}\dot{X}_1 &= f_1(X_1, X_2)\\ \dot{X}_2 &= f_2(X_1, X_2)\end{aligned}}$$

$$\begin{cases} f_1(X_{1o}, X_{2o}) = 0 \\ f_2(X_{1o}, X_{2o}) = 0 \end{cases} \longrightarrow$$

FIXED POINTS

PER CIASCUN PUNTO FISSO (X_{1o}, X_{2o}) **:**

Equazione Caratteristica

$$\lambda^2 - (f_{11} + f_{22})\lambda + (f_{11} f_{22} - f_{12} f_{21}) = 0$$

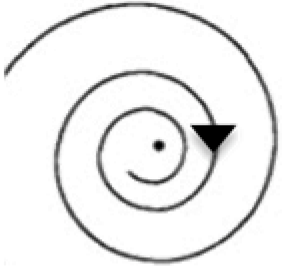
con $f_{ij} = \frac{\partial f_i}{\partial x_j}$

Le distanze del punto rappresentativo del sistema dal punto fisso lungo i due assi dello spazio degli stati spiraleggiano nei dintorni del punto:

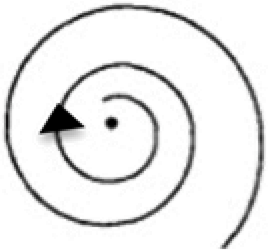
$$\boxed{\lambda_\pm = R \pm i\Omega}$$

$$\begin{cases} x_1(t) = F_1 e^{Rt} \sin \Omega t \\ x_2(t) = F_2 e^{Rt} \sin \Omega t \end{cases}$$

Ma cosa succede nel caso R = 0?

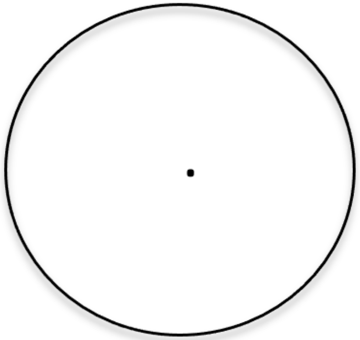


R<0: spiral node



R>0: spiral repellor

Nasce un CICLO LIMITE attorno al punto fisso!



R=0

Metodo dello Jacobiano per studiare i punti fissi nel caso generale a 2 dim.

Equazioni originarie

$$\dot{X}_1 = f_1(X_1 . X_2)$$
$$\dot{X}_2 = f_2(X_1 . X_2)$$

…ricavare i punti fissi…

Equazioni linearizzate nelle vicinanze di un dato punto fisso (X_{1o}, X_{2o})

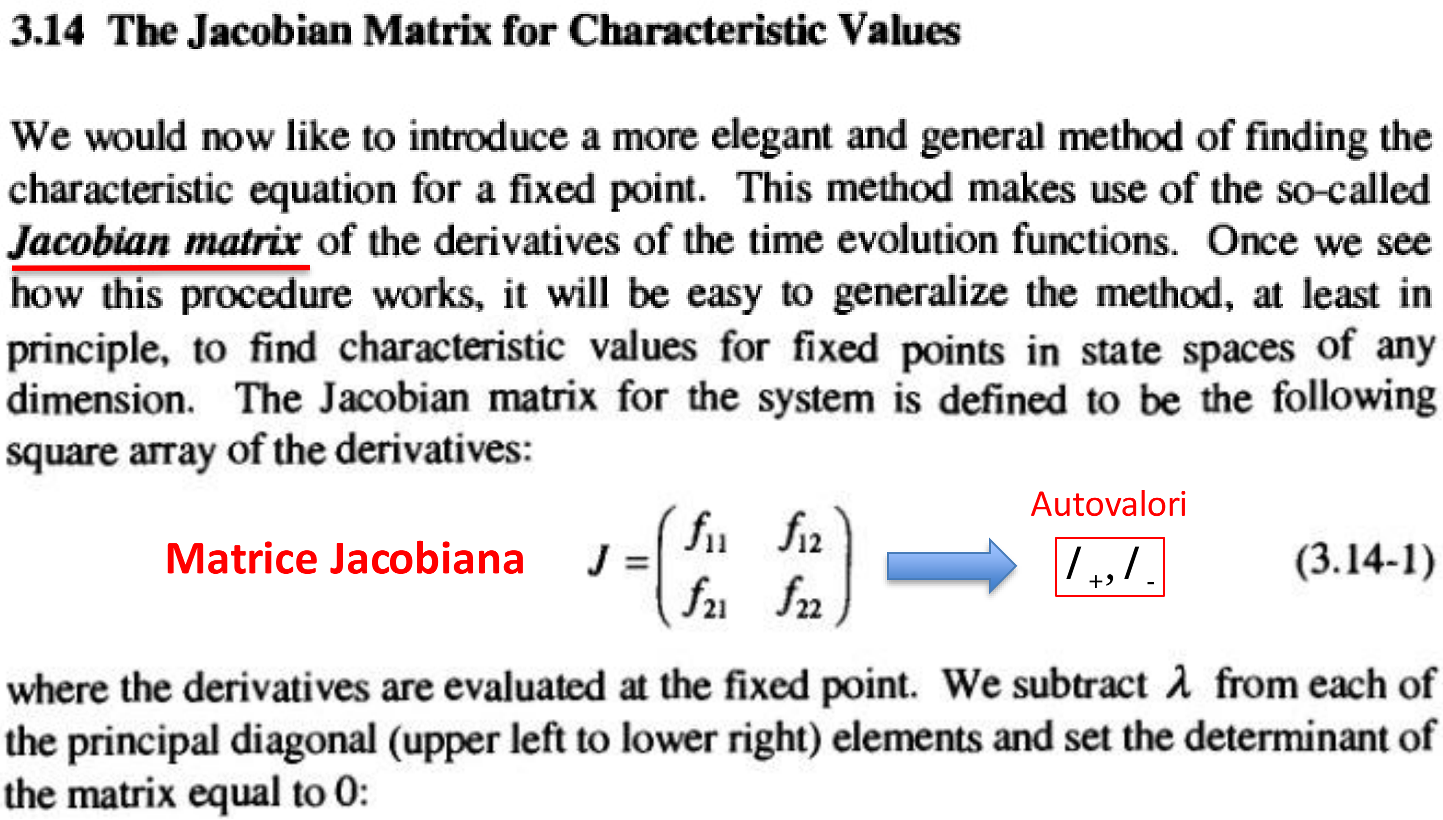
$$\dot{x}_1 = \frac{\partial f_1}{\partial x_1} x_1 + \frac{\partial f_1}{\partial x_2} x_2$$
$$\dot{x}_2 = \frac{\partial f_2}{\partial x_1} x_1 + \frac{\partial f_2}{\partial x_2} x_2$$

with $f_{ij} = \frac{\partial f_i}{\partial x_j}$

…calcolate nel punto fisso

Distanze dal punto fisso

$$x_1 = X_1 - X_{1o}$$
$$x_2 = X_2 - X_{2o}$$



3.14 The Jacobian Matrix for Characteristic Values

We would now like to introduce a more elegant and general method of finding the characteristic equation for a fixed point. This method makes use of the so-called ***Jacobian matrix*** of the derivatives of the time evolution functions. Once we see how this procedure works, it will be easy to generalize the method, at least in principle, to find characteristic values for fixed points in state spaces of any dimension. The Jacobian matrix for the system is defined to be the following square array of the derivatives:

Matrice Jacobiana

$$J = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \qquad (3.14\text{-}1)$$

Autovalori

$$\lambda_{+}, \lambda_{-}$$

where the derivatives are evaluated at the fixed point. We subtract λ from each of the principal diagonal (upper left to lower right) elements and set the determinant of the matrix equal to 0:

Metodo dello Jacobiano per studiare i punti fissi nel caso generale a 2 dim.

Eq. agli autovalori

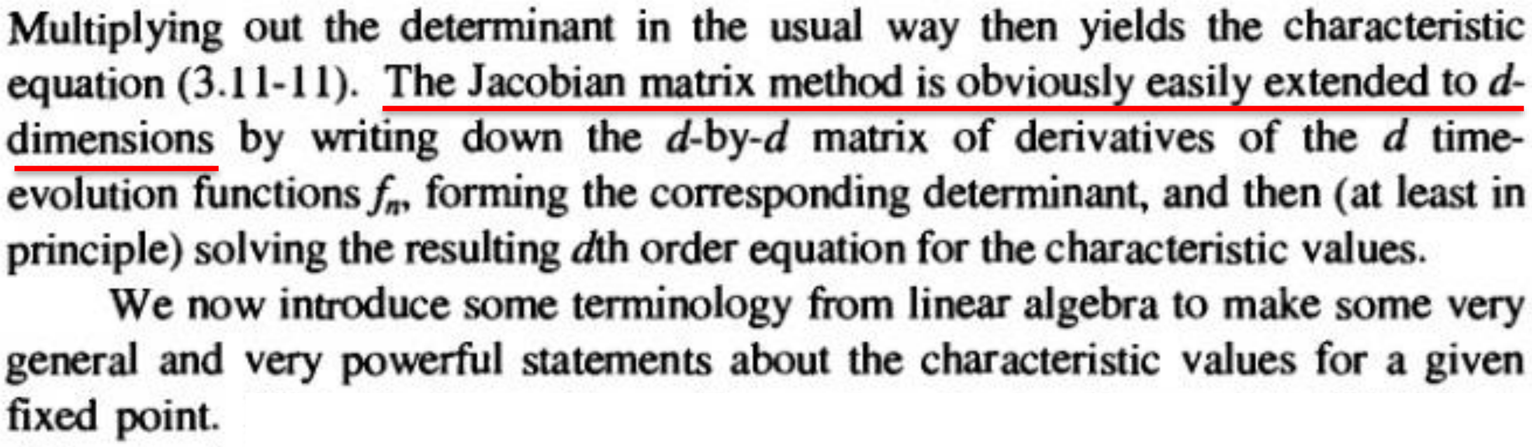
$$J = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \qquad \boxed{J\vec{v} = \lambda\vec{v}} \rightarrow \det(J - \lambda I) = 0 \rightarrow \begin{vmatrix} f_{11} - \lambda & f_{12} \\ f_{21} & f_{22} - \lambda \end{vmatrix} = 0$$

Equazione caratteristica dello Jacobiano

$$\lambda^2 - (f_{11} + f_{22})\lambda + (f_{11}f_{22} - f_{12}f_{21}) = 0 \tag{3.11-11}$$

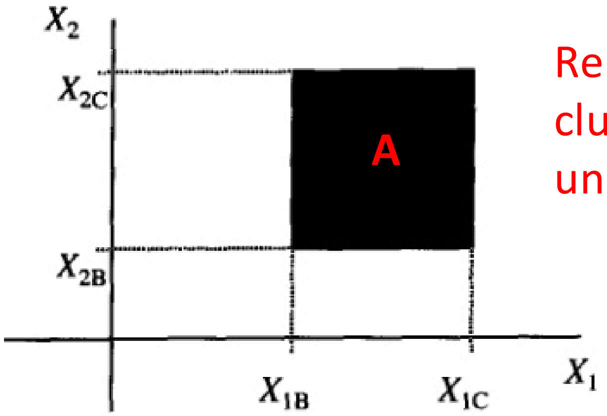
Autovalori dello Jacobiano

$$\lambda_{\pm} = \frac{f_{11} + f_{22} \pm \sqrt{(f_{11} + f_{22})^2 - 4(f_{11}f_{22} - f_{12}f_{21})}}{2} \tag{3.11-12}$$



Multiplying out the determinant in the usual way then yields the characteristic equation (3.11-11). The Jacobian matrix method is obviously easily extended to *d*-dimensions by writing down the *d*-by-*d* matrix of derivatives of the *d* time-evolution functions f_m, forming the corresponding determinant, and then (at least in principle) solving the resulting *d*th order equation for the characteristic values.

We now introduce some terminology from linear algebra to make some very general and very powerful statements about the characteristic values for a given fixed point.



Reminder: condizione affinchè un cluster di condizioni iniziali collassi su un attrattore stabile:

$$J = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

$$\frac{1}{A}\frac{dA}{dt} = (f_{11} + f_{22}) < 0 \longrightarrow \boxed{TrJ < 0}$$

First, the ***trace*** of a matrix, such as the Jacobian matrix (3.14-1), is defined to be the sum of the principal diagonal elements. For Eq. (3.14-1) this is explicitly

Traccia dello Jacobiano

$$TrJ = f_{11} + f_{22} \tag{3.14-3}$$

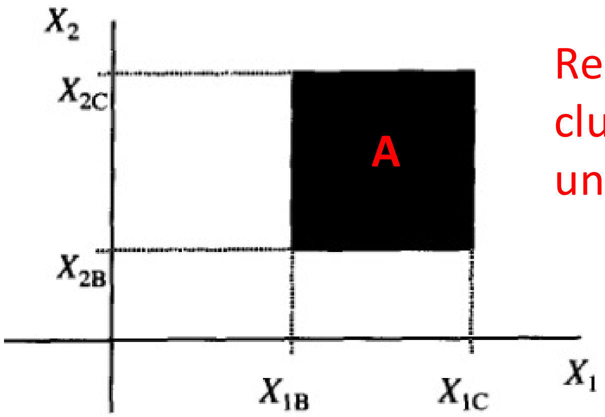
According to Eq. (3.13-5), however, this is just the combination of derivatives needed to test whether or not the system's trajectories collapse toward an attractor. To make a connection with the previous section, we note that $TrJ = 2R$, so that we see that the sign of TrJ determines whether the fixed point is a node or a repellor.

$$\boxed{\lambda_\pm = R}$$

$$\boxed{\lambda_\pm = R \pm i\Omega}$$

$$x_1(t) = F_1 e^{Rt} \sin Wt$$
$$x_2(t) = F_2 e^{Rt} \sin Wt$$

$$R = \frac{1}{2}(f_{11} + f_{22})$$



Reminder: condizione affinchè un cluster di condizioni iniziali collassi su un attrattore stabile:

$$J = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

$$\frac{1}{A}\frac{dA}{dt} = (f_{11} + f_{22}) < 0 \longrightarrow \boxed{TrJ < 0}$$

First, the ***trace*** of a matrix, such as the Jacobian matrix (3.14-1), is defined to be the sum of the principal diagonal elements. For Eq. (3.14-1) this is explicitly

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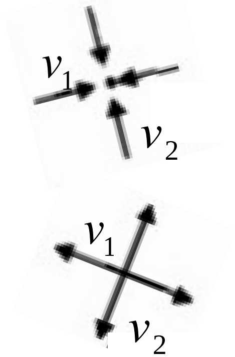
Linear algebra also tells us how to find the directions to be associated with the characteristic values.

In linear algebra this procedure is called "finding the eigenvalues and eigenvectors of the matrix." For our purposes, the eigenvalues are the characteristic values of the fixed point and the eigenvectors give the associated characteristic directions. However, we will not need these eigenvectors for most of our purposes. The interested reader is referred to the books on linear algebra listed at the end of the chapter.

Autovettori dello Jacobiano



$$J = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

We now introduce one more symbol:

Determinante dello Jacobiano: $\Delta = f_{11} f_{22} - f_{21} f_{12}$ (3.14-6)

Δ is called the ***determinant*** of that matrix. Then we may show that the nature of the fixed point is determined by *TrJ* and Δ as listed in Table 3.3.

$$\lambda_{\pm} = \frac{f_{11} + f_{22} \pm \sqrt{(f_{11} + f_{22})^2 - 4(f_{11} f_{22} - f_{12} f_{21})}}{2} \rightarrow \lambda_{\pm} = \frac{TrJ \pm \sqrt{(TrJ)^2 - 4\Delta}}{2}$$

$$\boxed{\lambda_{\pm} = R \pm i\Omega}$$

$$x_1(t) = F_1 e^{Rt} \sin \Omega t$$

$$x_2(t) = F_2 e^{Rt} \sin \Omega t$$

$$\begin{cases} R = \dfrac{1}{2} TrJ \\ \Omega = \dfrac{1}{2} \sqrt{TrJ^2 - 4\Delta} \end{cases}$$

Table 3.3

Fixed Points for Two-dimensional State Space

	$TrJ < 0$	$TrJ > 0$
$\Delta > (1/4)(TrJ)^2$	spiral node	spiral repellor
$0 < \Delta < (1/4)(TrJ)^2$	node	repellor
$\Delta < 0$	saddle point	saddle point

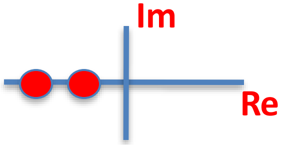
Riepilogo dei Punti Fissi in uno Spazio degli Stati a Due Dimensioni

$$\lambda_{\pm} = \frac{TrJ \pm \sqrt{(TrJ)^2 - 4D}}{2}$$

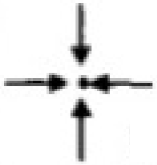
con:
$$\begin{cases} TrJ = f_{11} + f_{22} \\ D = f_{11}f_{22} - f_{21}f_{12} \end{cases}$$

$D > 0$

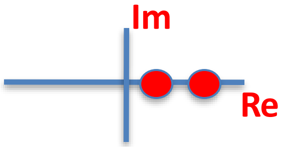
- $\boxed{(TrJ)^2 - 4D > 0}$
  $0 < D < \frac{1}{4}(TrJ)^2 \rightarrow \lambda_{+}, \lambda_{-}$ **reali e concordi**
  - $TrJ < 0$



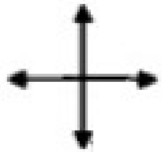
NODE



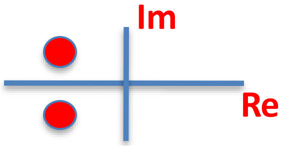
  - $TrJ > 0$



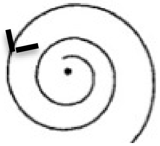
REPELLOR



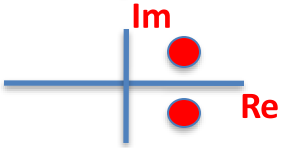
- $\boxed{(TrJ)^2 - 4D < 0}$
  $D > \frac{1}{4}(TrJ)^2 \rightarrow \lambda_{+}, \lambda_{-}$ **complessi coniugati**
  - $TrJ < 0$



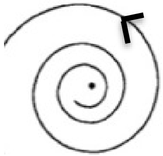
SPIRAL NODE



  - $TrJ > 0$

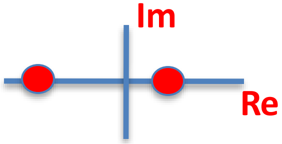


SPIRAL REPELLOR

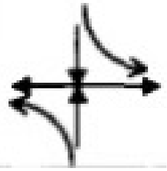


$D < 0 \rightarrow \lambda_{+}, \lambda_{-}$ **reali e discordi**

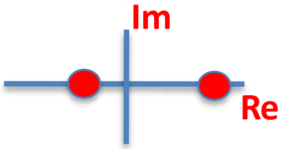
- $TrJ < 0$



SADDLE POINT



- $TrJ > 0$



SADDLE POINT

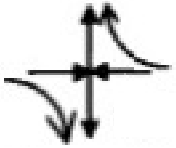


Diagramma dei Punti Fissi in uno Spazio degli Stati a Due Dimensioni

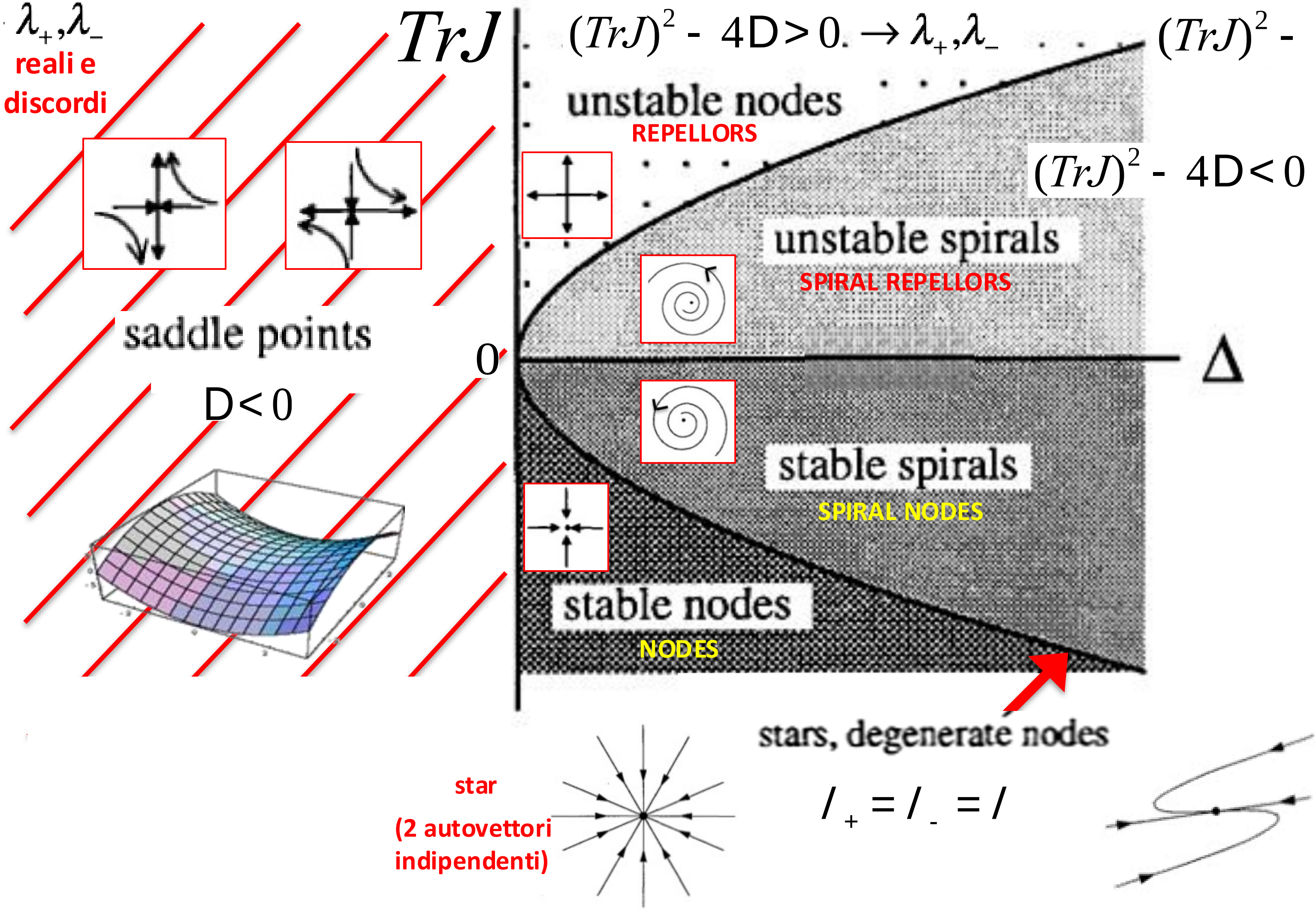
$$\lambda_\pm = \frac{TrJ \pm \sqrt{(TrJ)^2 - 4\Delta}}{2}$$

con: $\begin{cases} TrJ = f_{11} + f_{22} \\ \Delta = f_{11}f_{22} - f_{21}f_{12} \end{cases}$

$(TrJ)^2 - 4\Delta > 0$

λ_+, λ_-

reali e discordi



reali e concordi

$(TrJ)^2 - 4\Delta > 0 \to \lambda_+, \lambda_-$

$(TrJ)^2 - 4\Delta = 0$

$(TrJ)^2 - 4\Delta < 0 \to \lambda_+, \lambda_-$

complessi coniugati

TrJ
unstable nodes
REPELLORS
unstable spirals
SPIRAL REPELLORS
saddle points
0
Δ
stable spirals
SPIRAL NODES
stable nodes
NODES
stars, degenerate nodes

$\Delta < 0$

star

(2 autovettori indipendenti)

$\lambda_+ = \lambda_- = \lambda$

degenerate node

(1 autovettore)

Summary of Fixed Point Analysis for Two-dimensional State Space

1. Write the time evolution equations in the first-order time derivative form of Eq. (3.10-1).

$$\begin{aligned}\dot{X}_1 &= f_1(X_1, X_2)\\ \dot{X}_2 &= f_2(X_1, X_2)\end{aligned} \tag{3.10-1}$$

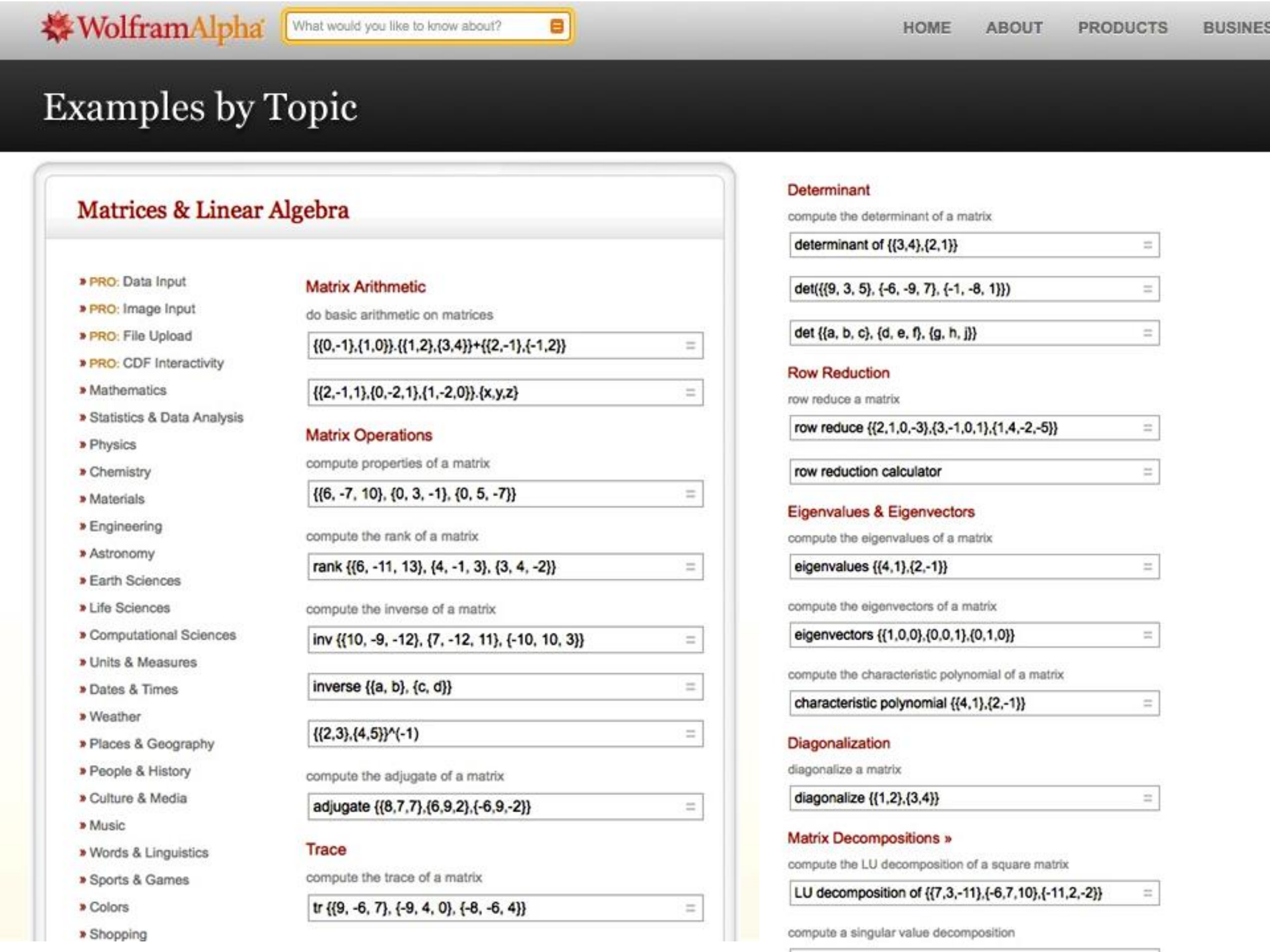
2. Find the fixed points of the evolution by finding those points that satisfy

$$\begin{aligned}f_1(X_1, X_2) &= 0\\ f_2(X_1, X_2) &= 0\end{aligned}$$

3. At the fixed points, evaluate the partial derivatives of the time evolution functions to set up the Jacobian matrix

$$J \equiv \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \tag{3.14-1}$$

4. Evaluate the trace and determinant of the Jacobian matrix at the fixed point and use Table 3.3 to find the type of fixed point.
5. Use Eq. (3.11-12) to find the numerical values of the characteristic values and to specify the behavior of the state-space trajectories near the fixed point with Eq. (3.11-13).



Examples by Topic

Matrices & Linear Algebra

- PRO: Data Input
- PRO: Image Input
- PRO: File Upload
- PRO: CDF Interactivity
- Mathematics
- Statistics & Data Analysis
- Physics
- Chemistry
- Materials
- Engineering
- Astronomy
- Earth Sciences
- Life Sciences
- Computational Sciences
- Units & Measures
- Dates & Times
- Weather
- Places & Geography
- People & History
- Culture & Media
- Music
- Words & Linguistics
- Sports & Games
- Colors
- Shopping

Matrix Arithmetic

do basic arithmetic on matrices

{{0,-1},{1,0}}.{{1,2},{3,4}}+{{2,-1},{-1,2}}

{{2,-1,1},{0,-2,1},{1,-2,0}}.{x,y,z}

Matrix Operations

compute properties of a matrix

{{6, -7, 10}, {0, 3, -1}, {0, 5, -7}}

compute the rank of a matrix

rank {{6, -11, 13}, {4, -1, 3}, {3, 4, -2}}

compute the inverse of a matrix

inv {{10, -9, -12}, {7, -12, 11}, {-10, 10, 3}}

inverse {{a, b}, {c, d}}

{{2,3},{4,5}}^(-1)

compute the adjugate of a matrix

adjugate {{8,7,7},{6,9,2},{-6,9,-2}}

Trace

compute the trace of a matrix

tr {{9, -6, 7}, {-9, 4, 0}, {-8, -6, 4}}

Determinant

compute the determinant of a matrix

determinant of {{3,4},{2,1}}

det({{9, 3, 5}, {-6, -9, 7}, {-1, -8, 1}})

det {{a, b, c}, {d, e, f}, {g, h, i}}

Row Reduction

row reduce a matrix

row reduce {{2,1,0,-3},{3,-1,0,1},{1,4,-2,-5}}

row reduction calculator

Eigenvalues & Eigenvectors

compute the eigenvalues of a matrix

eigenvalues {{4,1},{2,-1}}

compute the eigenvectors of a matrix

eigenvectors {{1,0,0},{0,0,1},{0,1,0}}

compute the characteristic polynomial of a matrix

characteristic polynomial {{4,1},{2,-1}}

Diagonalization

diagonalize a matrix

diagonalize {{1,2},{3,4}}

Matrix Decompositions »

compute the LU decomposition of a square matrix

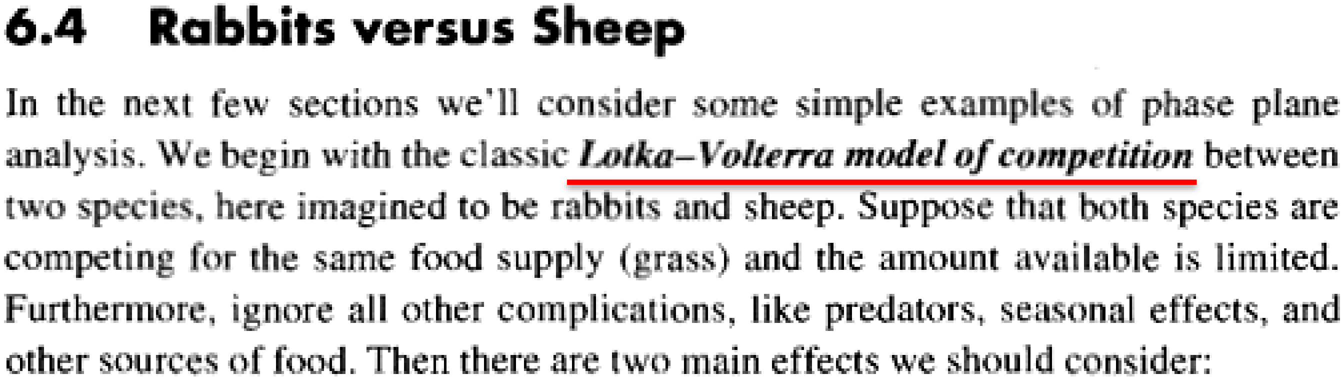
LU decomposition of {{7,3,-11},{-6,7,10},{-11,2,-2}}

compute a singular value decomposition

Rabbit

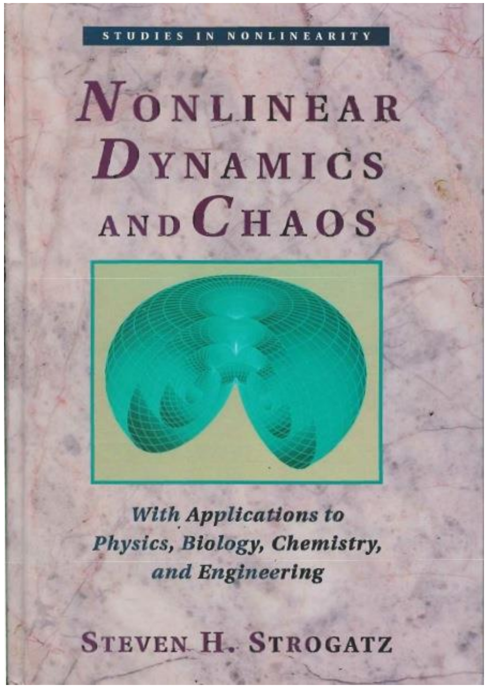
Steven
Strogatz

Sheep



6.4 Rabbits versus Sheep

In the next few sections we'll consider some simple examples of phase plane analysis. We begin with the classic ***Lotka–Volterra model of competition*** between two species, here imagined to be rabbits and sheep. Suppose that both species are competing for the same food supply (grass) and the amount available is limited. Furthermore, ignore all other complications, like predators, seasonal effects, and other sources of food. Then there are two main effects we should consider:

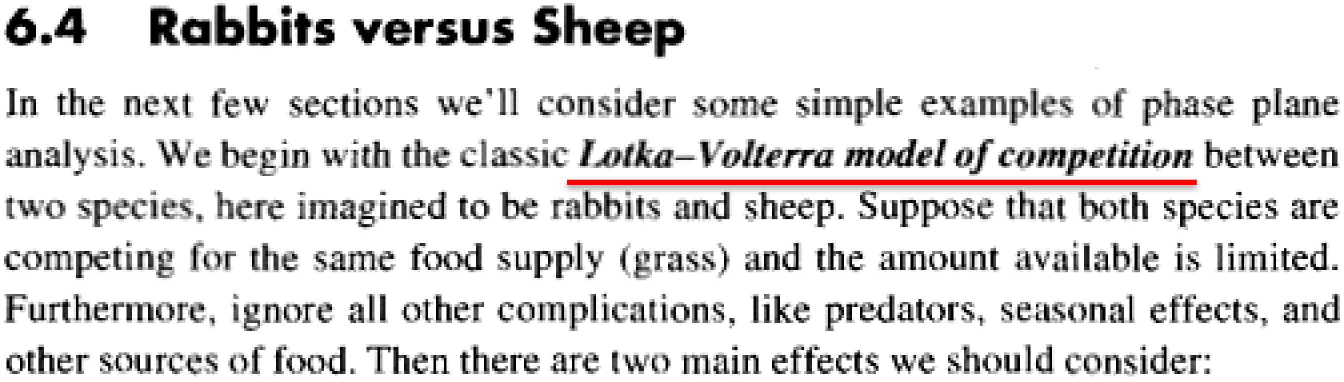


STUDIES IN NONLINEARITY
NONLINEAR DYNAMICS AND CHAOS
With Applications to Physics, Biology, Chemistry, and Engineering
STEVEN H. STROGATZ

Rabbit

Steven Strogatz

Sheep



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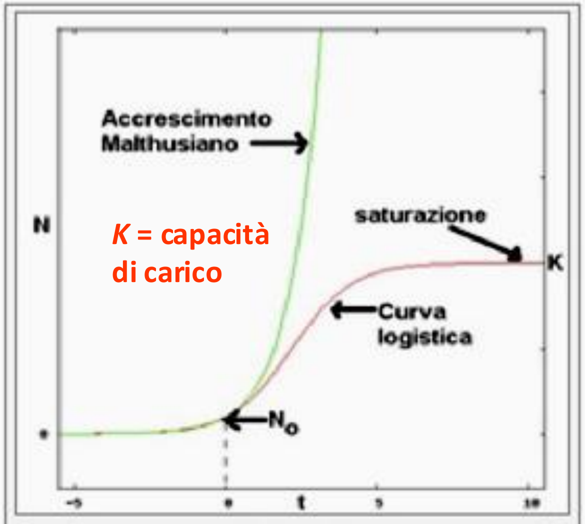
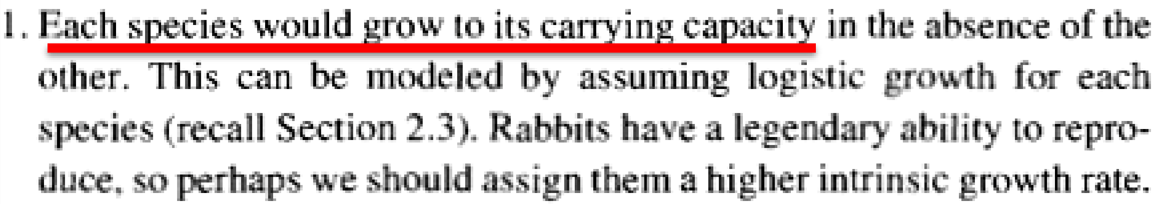


Confronto tra curva logistica e curva di accrescimento esponenziale (malthusiano). I parametri sono: $k = 10, N_0 = 1, r = 1$



1. Each species would grow to its carrying capacity in the absence of the other. This can be modeled by assuming logistic growth for each species (recall Section 2.3). Rabbits have a legendary ability to reproduce, so perhaps we should assign them a higher intrinsic growth rate.

Steven Strogatz

Rabbit

Sheep

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1. Each species would grow to its carrying capacity in the absence of the other. This can be modeled by assuming logistic growth for each species (recall Section 2.3). Rabbits have a legendary ability to reproduce, so perhaps we should assign them a higher intrinsic growth rate.

2. When rabbits and sheep encounter each other, trouble starts. Sometimes the rabbit gets to eat, but more usually the sheep nudges the rabbit aside and starts nibbling (on the grass, that is). We'll assume that these conflicts occur at a rate proportional to the size of each population. (If there were twice as many sheep, the odds of a rabbit encountering a sheep would be twice as great.) Furthermore, we assume that the conflicts reduce the growth rate for each species, but the effect is more severe for the rabbits.

$$x = 0 \rightarrow \dot{x} = 0$$

$$\dot{y} = 2y(1 - \frac{y}{2})$$

$$[b = 1, a = K = 2]$$

A specific model that incorporates these assumptions is

Rabbit

$$\begin{cases} \dot{x} = x(3 - x - 2y) \\ \dot{y} = y(2 - x - y) \end{cases}$$

Sheep

where

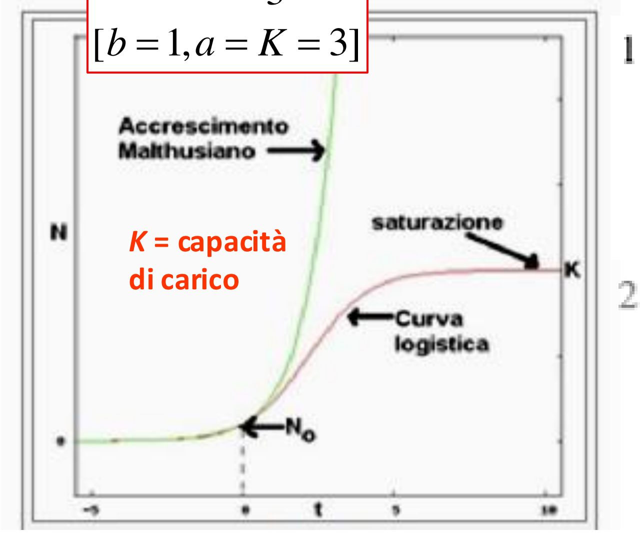
$x(t) =$ population of rabbits,
$y(t) =$ population of sheep

and $x, y \geq 0$. The coefficients have been chosen to reflect this scenario, but are otherwise arbitrary. In the exercises, you'll be asked to study what happens if the coefficients are changed.

$$y = 0 \rightarrow \dot{y} = 0$$

$$\dot{x} = 3x(1 - \frac{x}{3})$$

$$[b = 1, a = K = 3]$$



$$\boxed{\frac{dN}{dt} = aN\left(1 - \frac{N}{K}\right)}$$

con $K = \frac{a}{b}$

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A specific model that incorporates these assumptions is

$$\begin{cases} \dot{x} = x(3 - x \boxed{- 2y}) \\ \dot{y} = y(2 \boxed{- x} - y) \end{cases}$$

Rabbit

Sheep

where

$x(t) =$ population of rabbits,
$y(t) =$ population of sheep

$y = 0 \rightarrow \dot{y} = 0$

$\dot{x} = 3x(1 - \frac{x}{3})$

$[b = 1, a = K = 3]$

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where

$x(t) =$ population of rabbits,
$y(t) =$ population of sheep

Sheep

$x = 0 \rightarrow \dot{x} = 0$

$\dot{y} = 2y(1 - \frac{y}{2})$

$[b = 1, a = K = 2]$

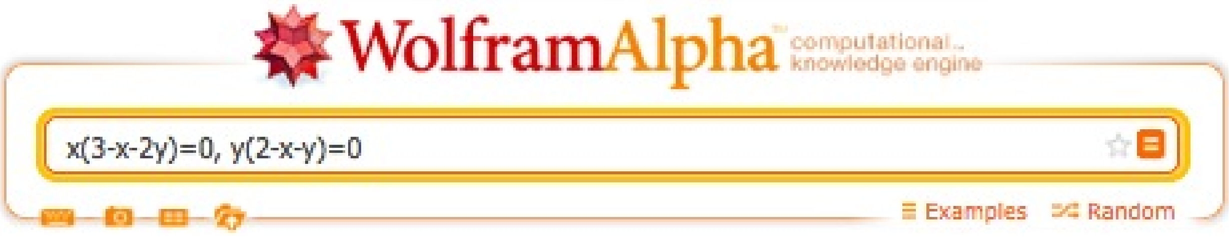
$y = 0 \rightarrow \dot{y} = 0$

$\dot{x} = 3x(1 - \frac{x}{3})$

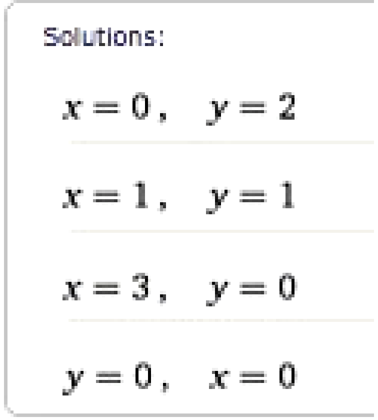
$[b = 1, a = K = 3]$

and $x, y \geq 0$. The coefficients have been chosen to reflect this scenario, but are otherwise arbitrary. In the exercises, you'll be asked to study what happens if the coefficients are changed.

To find the fixed points for the system, we solve $\dot{x} = 0$ and $\dot{y} = 0$ simultaneously. Four fixed points are obtained: (0,0), (0,2), (3,0), and (1,1).



x(3-x-2y)=0, y(2-x-y)=0



Solutions:

$x = 0, \quad y = 2$

$x = 1, \quad y = 1$

$x = 3, \quad y = 0$

$y = 0, \quad x = 0$

$$x = 0 \rightarrow \dot{x} = 0$$

$$\dot{y} = 2y(1-\frac{y}{2})$$

$$[b = 1, a = K = 2]$$

A specific model that incorporates these assumptions is

Rabbit

$$\begin{cases} \dot{x} = x(3 - x - 2y) \\ \dot{y} = y(2 - x - y) \end{cases}$$

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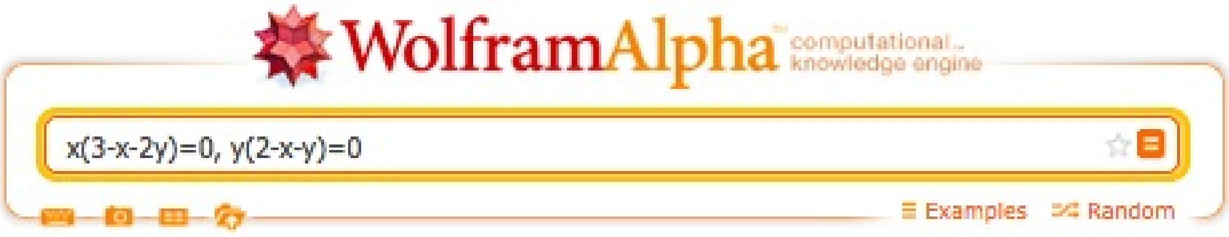
$$[b = 1, a = K = 3]$$

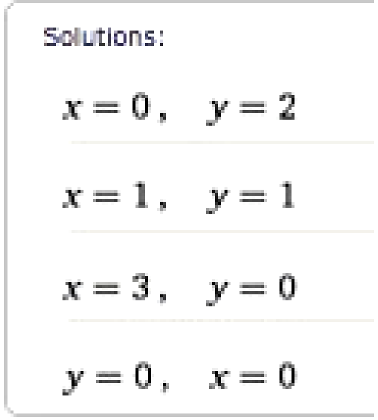
and $x, y \geq 0$. The coefficients have been chosen to reflect this scenario, but are otherwise arbitrary. In the exercises, you'll be asked to study what happens if the coefficients are changed.

To find the fixed points for the system, we solve $\dot{x} = 0$ and $\dot{y} = 0$ simultaneously. Four fixed points are obtained: $(0,0)$, $(0,2)$, $(3,0)$, and $(1,1)$. To classify them, we compute the Jacobian:

$$J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}.$$



x(3-x-2y)=0, y(2-x-y)=0



Solutions:

$x = 0, \quad y = 2$

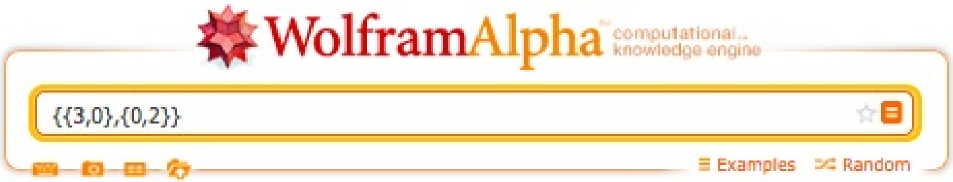
$x = 1, \quad y = 1$

$x = 3, \quad y = 0$

$y = 0, \quad x = 0$

Now consider the four fixed points in turn:

$$(\underline{0,0}):\ \text{Then } J=\begin{pmatrix}3 & 0\\ 0 & 2\end{pmatrix}.$$


WolframAlpha computational knowledge engine
Examples
Random

{{3,0},{0,2}}

The eigenvalues are $\lambda = 3\,,\,2$ so $(0,0)$ is an *unstable node*. Trajectories leave the origin parallel to the eigenvector for $\lambda = 2$, i.e. tangential to $\mathbf{v} = (0,1)$, which spans the y-axis. (Recall the general rule: at a node, trajectories are tangential to the slow eigendirection, which is the eigendirection with the smallest $|\lambda|$.) Thus, the phase portrait near $(0,0)$ looks like Figure 6.4.1.

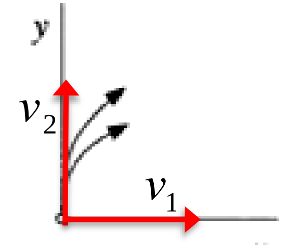
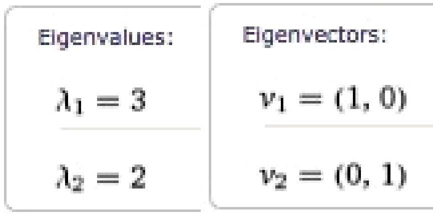


Figure 6.4.1

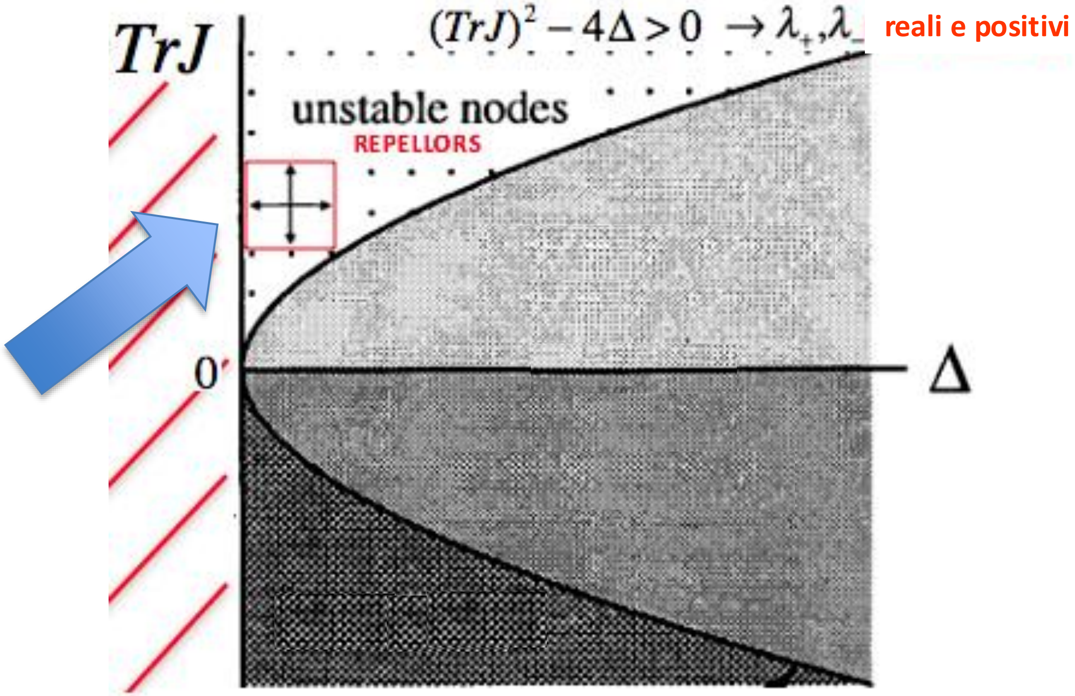


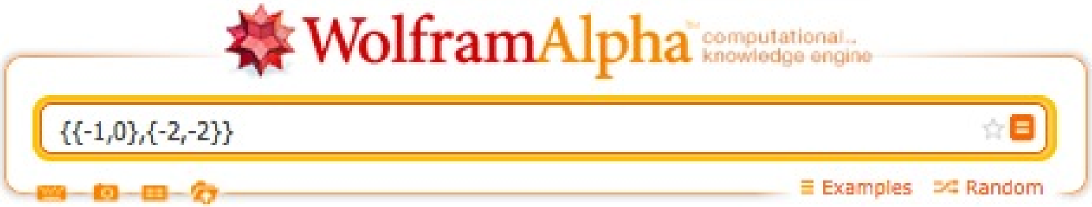
Eigenvalues:	Eigenvectors:
$\lambda_1 = 3$	$v_1 = (1, 0)$
$\lambda_2 = 2$	$v_2 = (0, 1)$

$$D = 6 > 0$$

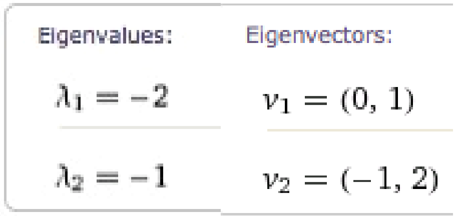
$$TrJ = 5 > 0$$

$$(TrJ)^2 - 4D = 1 > 0$$


TrJ
$(TrJ)^2 - 4\Delta > 0 \rightarrow \lambda_+, \lambda_-$
reali e positivi
unstable nodes
REPELLORS
0
Δ


WolframAlpha computational knowledge engine
Examples
Random

{{-1,0},{-2,-2}}



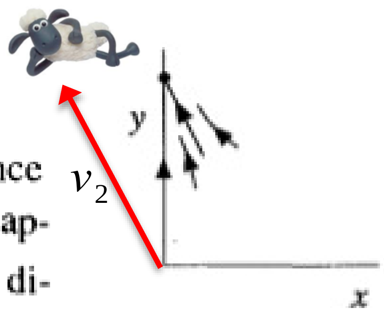
Eigenvalues:	Eigenvectors:
$\lambda_1 = -2$	$v_1 = (0, 1)$
$\lambda_2 = -1$	$v_2 = (-1, 2)$

(0,2): Then $J = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$.

$D = 2 > 0$

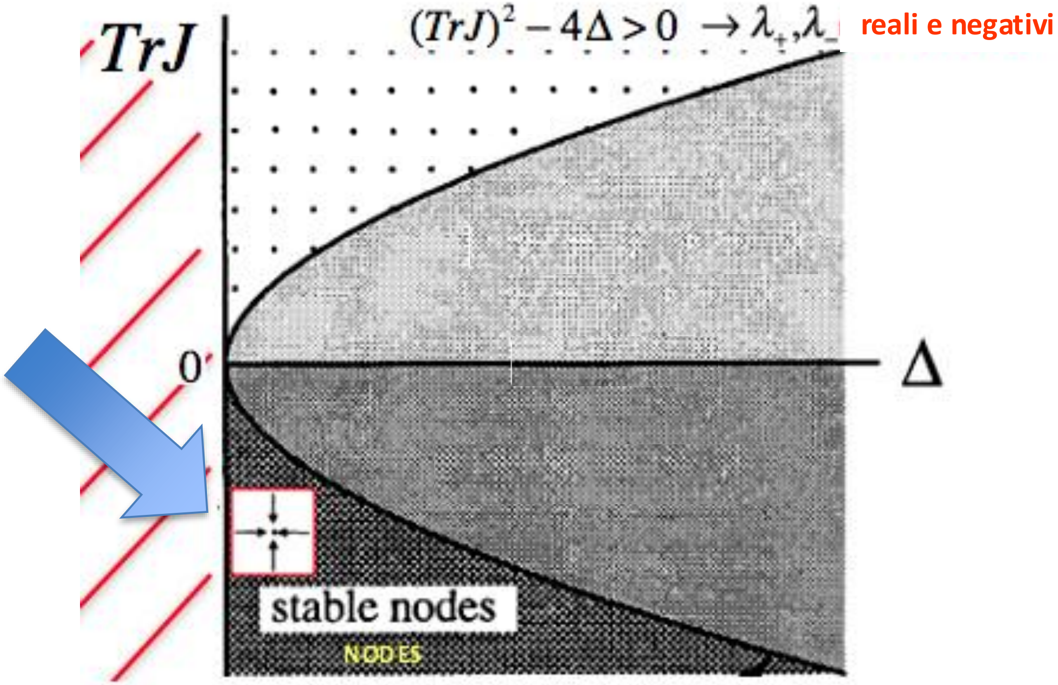
$TrJ = -3 < 0$

$(TrJ)^2 - 4D = 1 > 0$



This matrix has eigenvalues $\lambda = -1, -2$, as can be seen from inspection, since the matrix is triangular. Hence the fixed point is a *stable node*. Trajectories approach along the eigendirection associated with $\lambda = -1$; you can check that this direction is spanned by $\mathbf{v} = (-1, 2)$. Figure 6.4.2 shows the phase portrait near the fixed point (0, 2).

Figure 6.4.2


$(TrJ)^2 - 4\Delta > 0 \rightarrow \lambda_+, \lambda_-$
reali e negativi
TrJ
0
Δ
stable nodes
NODES

$(3,0)$: Then $J = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$ and $\lambda = -3, -1$.

This is also a *stable node*. The trajectories approach along the slow eigendirection spanned by $\mathbf{v} = (3, -1)$, as shown in Figure 6.4.3.

$$D = 3 > 0$$

$$TrJ = -4 < 0$$

$$(TrJ)^2 - 4D = 4 > 0$$

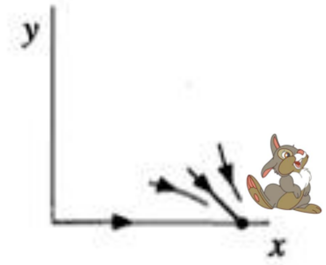
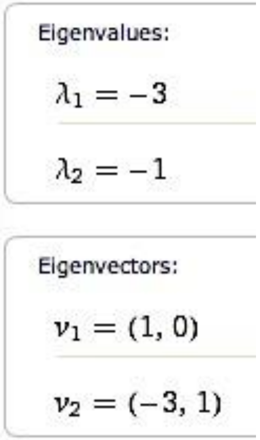


Figure 6.4.3



Eigenvalues:

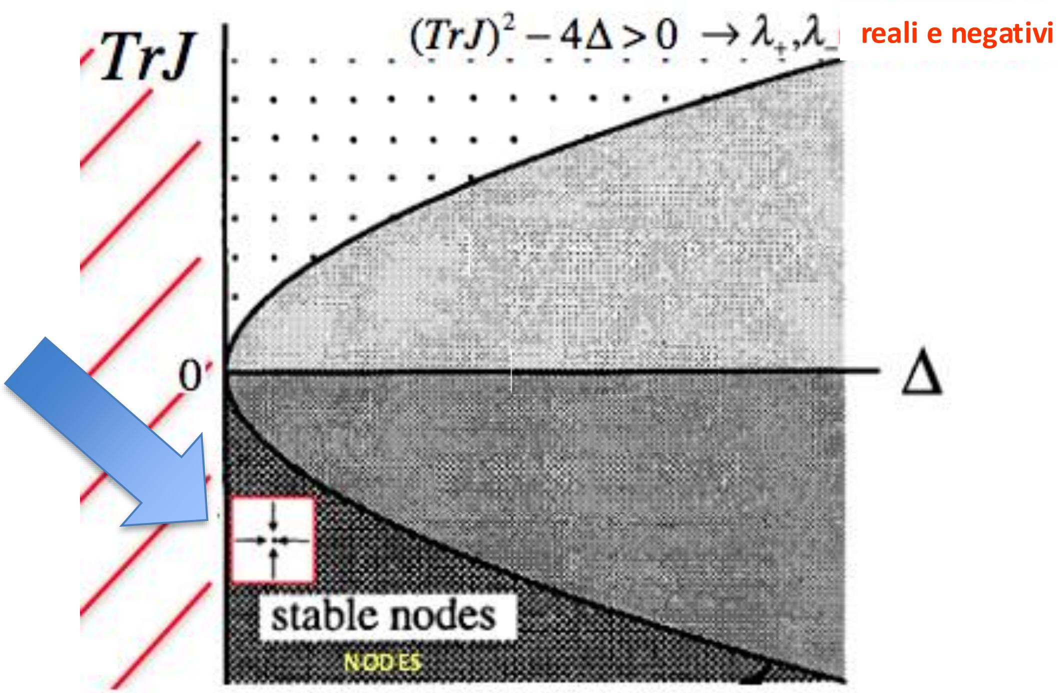
$\lambda_1 = -3$

$\lambda_2 = -1$

Eigenvectors:

$v_1 = (1, 0)$

$v_2 = (-3, 1)$



$(TrJ)^2 - 4\Delta > 0 \rightarrow \lambda_+, \lambda_-$ reali e negativi

v_2

(1,1): Then $J = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$, which has $\tau = -2$, $\Delta = -1$, and $\lambda = -1 \pm \sqrt{2}$.

Hence this is a *saddle point*. As you can check, the phase portrait near (1,1) is as shown in Figure 6.4.4.

$D = -1 < 0$

$TrJ = -2 < 0$

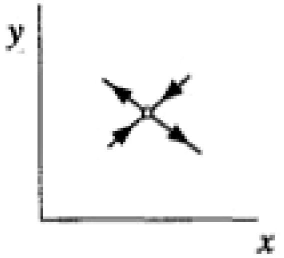
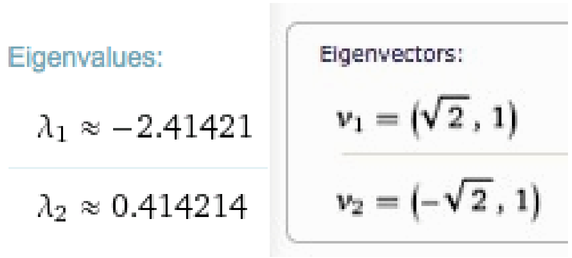


Figure 6.4.4



Eigenvalues:

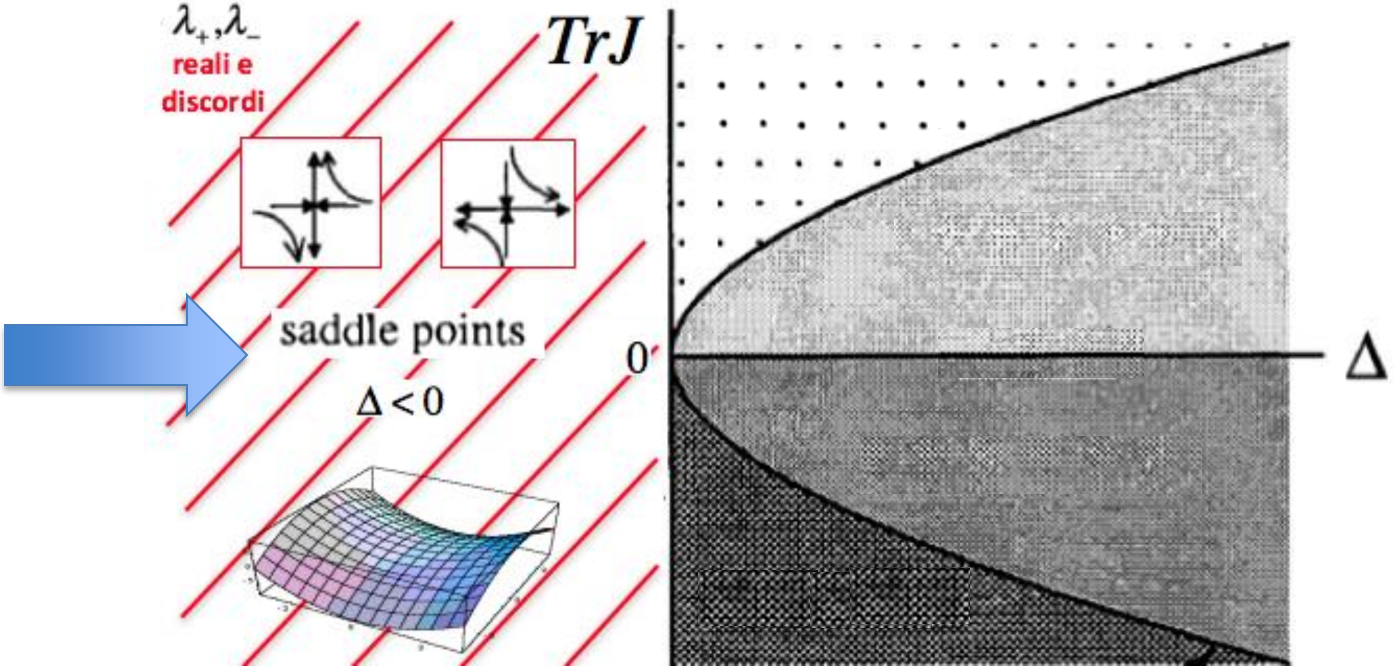
$\lambda_1 \approx -2.41421$

$\lambda_2 \approx 0.414214$

Eigenvectors:

$v_1 = (\sqrt{2}, 1)$

$v_2 = (-\sqrt{2}, 1)$


λ_+, λ_-
reali e discordi
TrJ
saddle points
$\Delta < 0$
0
Δ

Combining Figures 6.4.1–6.4.4, we get Figure 6.4.5, which already conveys a good sense of the entire phase portrait. Furthermore, notice that the x and y axes contain straight-line trajectories, since $\dot{x}=0$ when $x=0$, and $\dot{y}=0$ when $y=0$.

$$\begin{cases} \dot{x} = x(3-x-2y) \\ \dot{y} = y(2-x-y) \end{cases}$$

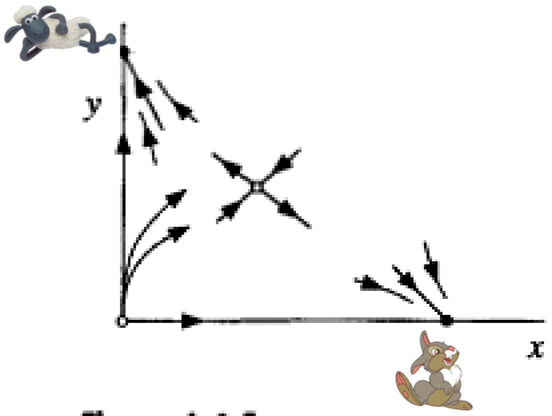
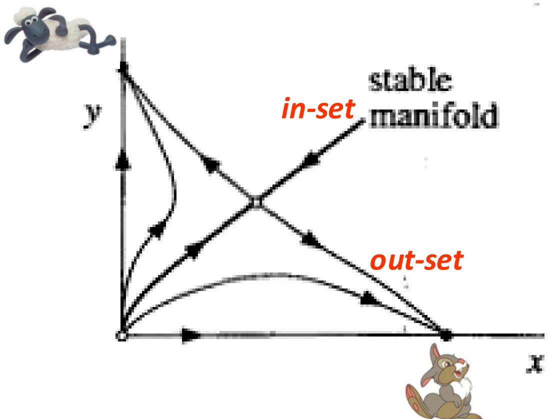


Figure 6.4.5

Now we use common sense to fill in the rest of the phase portrait (Figure 6.4.6). For example, some of the trajectories starting near the origin must go to the stable node on the x-axis, while others must go to the stable node on the y-axis. In between, there must be a special trajectory that can't decide which way to turn, and so it dives into the saddle point. This trajectory is part of the ***stable manifold*** of the saddle, drawn with a heavy line in Figure 6.4.6.



Ritratto globale nello spazio degli stati

The other branch of the stable manifold consists of a trajectory coming in "from infinity." A computer-generated phase portrait (Figure 6.4.7) confirms our sketch.

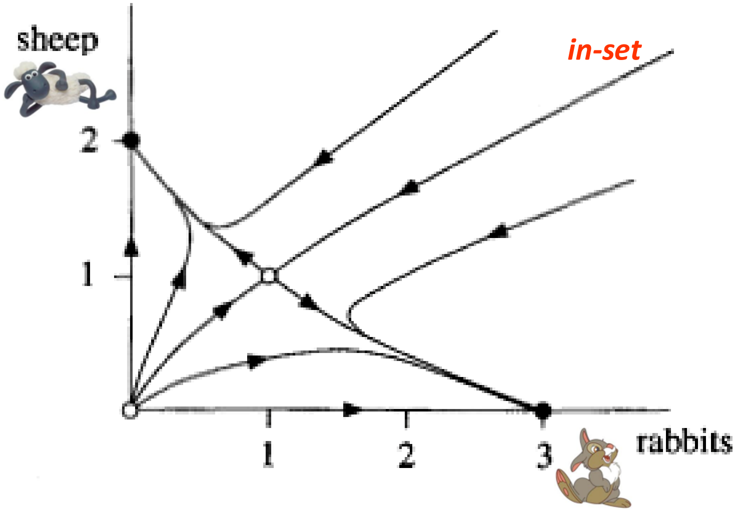
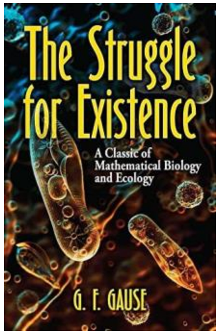


Figure 6.4.7

The phase portrait has an interesting biological interpretation. It shows that one species generally drives the other to extinction. Trajectories starting below the stable manifold lead to eventual extinction of the sheep, while those starting above lead to eventual extinction of the rabbits. This dichotomy occurs in other models of competition and has led biologists to formulate the *principle of competitive exclusion*, which states that two species competing for the same limited resource typically cannot coexist.



The Struggle for Existence

A Classic of Mathematical Biology and Ecology

G. F. GAUSE

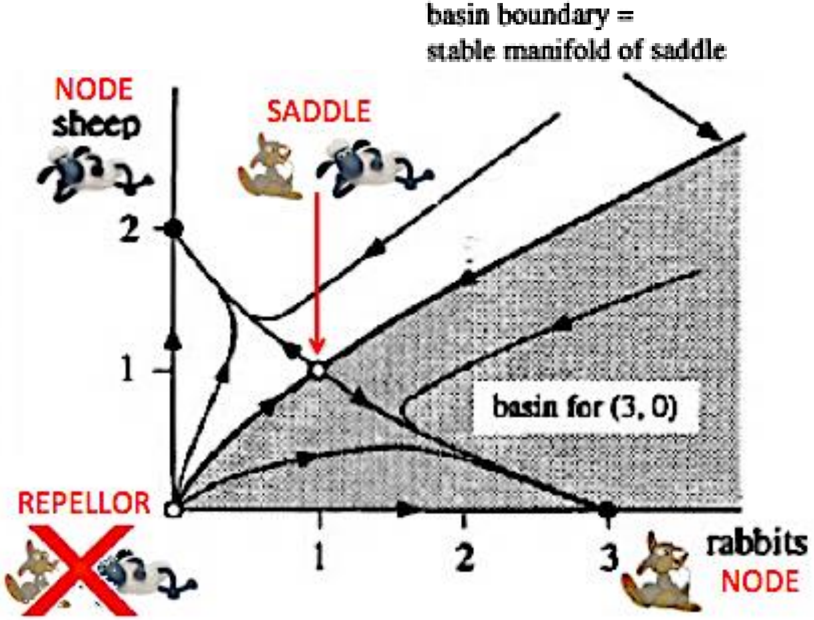
[illegible]

[illegible]

[illegible]

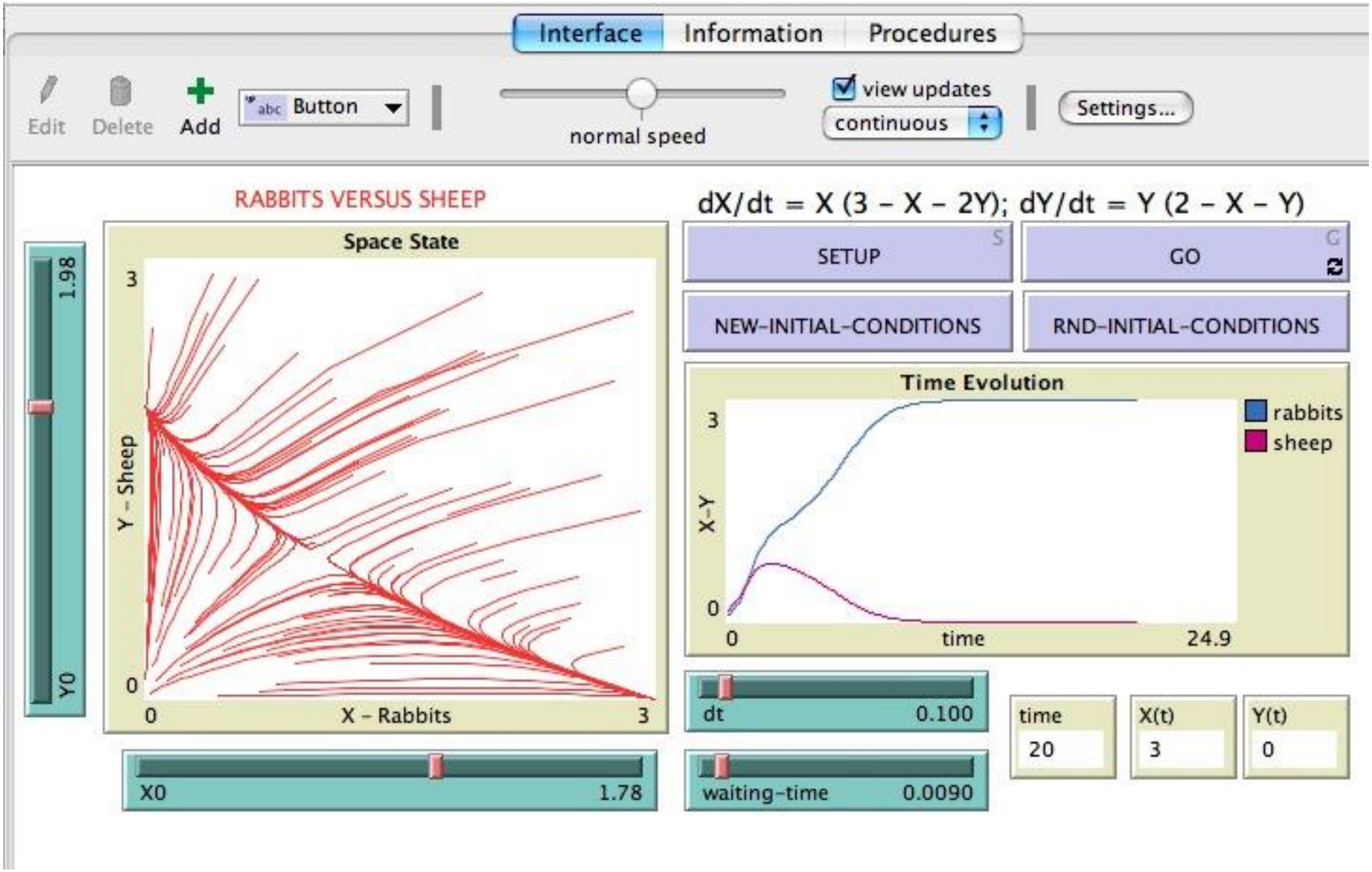
[illegible]

Our example also illustrates some general mathematical concepts. Given an attracting fixed point $\mathbf{x}^*$, we define its ***basin of attraction*** to be the set of initial conditions $\mathbf{x}_0$ such that $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$. For instance, the basin of attraction for the node at $(3,0)$ consists of all the points lying below the stable manifold of the saddle. This basin is shown as the shaded region in Figure 6.4.8.


basin boundary =
stable manifold of saddle
NODE
sheep
SADDLE
2
1
basin for (3, 0)
REPELLOR
1
2
3
rabbits
NODE

Because the stable manifold separates the basins for the two nodes, it is called the ***basin boundary***. For the same reason, the two trajectories that comprise the stable manifold are traditionally called ***separatrices***. Basins and their boundaries are important because they partition the phase space into regions of different long-term behavior.

rabbits_sheep.nlogo



RABBITS VERSUS SHEEP

$dX/dt = X(3 - X - 2Y); \; dY/dt = Y(2 - X - Y)$

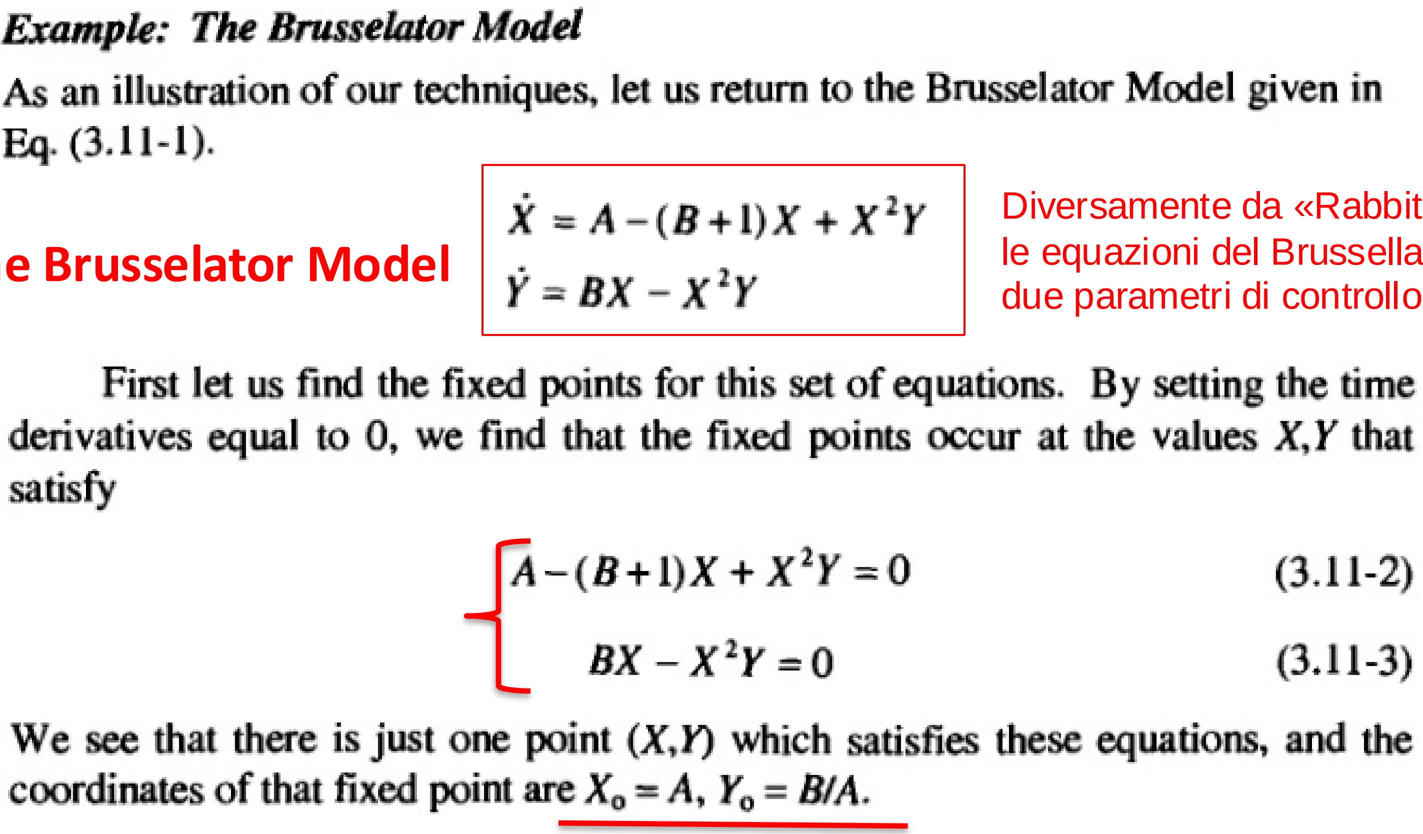
☑ view updates
continuous

Y0 1.98
X0 1.78
dt 0.100
waiting-time 0.0090

time 20
X(t) 3
Y(t) 0

Interface
Information
Procedures
Edit
Delete
Add
Button
normal speed
Settings...
SETUP
S
GO
G
NEW-INITIAL-CONDITIONS
RND-INITIAL-CONDITIONS
Space State
3
Y - Sheep
0
0
X - Rabbits
3
Time Evolution
3
X-Y
0
0
time
24.9
rabbits
sheep

Sistema dinamico con due parametri di controllo e punti fissi con autovalori complessi coniugati



Example: The Brusselator Model

As an illustration of our techniques, let us return to the Brusselator Model given in Eq. (3.11-1).

The Brusselator Model

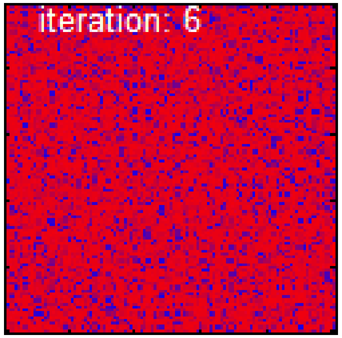
$$\boxed{\begin{aligned}\dot{X} &= A-(B+1)X+X^2Y \\ \dot{Y} &= BX-X^2Y\end{aligned}}$$

Diversamente da «Rabbits vs Sheep», le equazioni del Brussellators hanno due parametri di controllo A e B

First let us find the fixed points for this set of equations. By setting the time derivatives equal to 0, we find that the fixed points occur at the values X,Y that satisfy

$$\left\{\begin{aligned} A-(B+1)X+X^2Y &= 0 \qquad (3.11\text{-}2) \\ BX-X^2Y &= 0 \qquad (3.11\text{-}3) \end{aligned}\right.$$

We see that there is just one point (X,Y) which satisfies these equations, and the coordinates of that fixed point are $\underline{X_0 = A,\ Y_0 = B/A}$.



Simulation of the Brusselator as reaction-diffusion system in two spatial dimensions

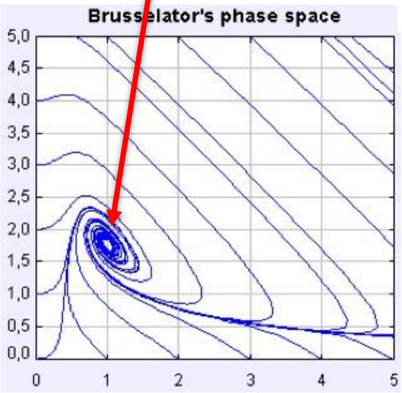
Ilya Prigogine
(1917-2003)

Sistema dinamico con due parametri di controllo e punti fissi con autovalori complessi coniugati

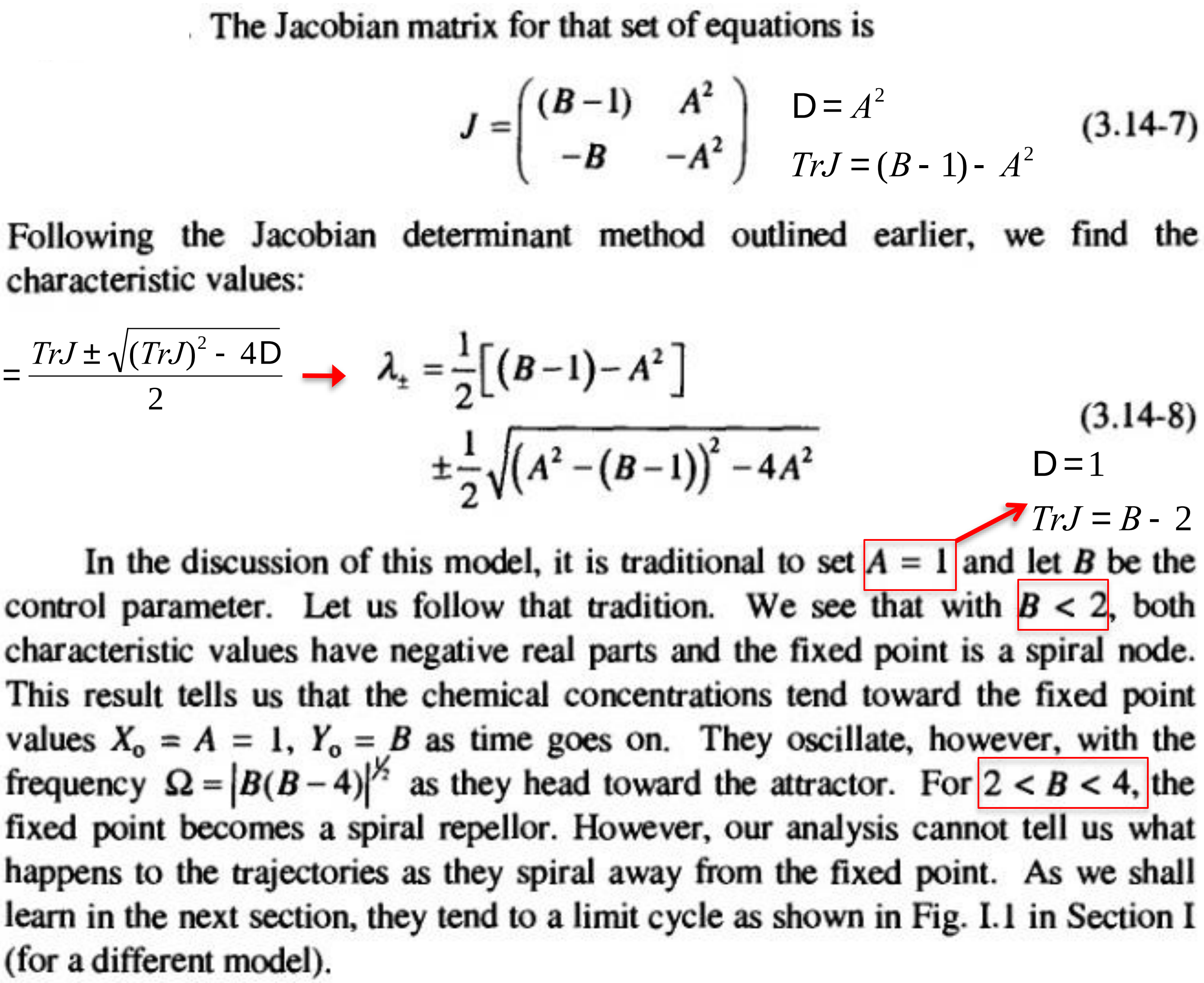
$$\dot{X} = A-(B+1)X+X^2Y$$
$$\dot{Y} = BX-X^2Y$$

1 punto fisso:

$X_0 = A,\ Y_0 = B/A.$



Ilya Prigogine
(1917-2003)



The Jacobian matrix for that set of equations is

$$J = \begin{pmatrix} (B-1) & A^2 \\ -B & -A^2 \end{pmatrix} \qquad (3.14\text{-}7)$$

D = A^2

$TrJ = (B-1) - A^2$

Following the Jacobian determinant method outlined earlier, we find the characteristic values:

$$l_\pm = \frac{TrJ \pm \sqrt{(TrJ)^2 - 4D}}{2} \rightarrow \lambda_\pm = \frac{1}{2}\left[(B-1)-A^2\right] \pm \frac{1}{2}\sqrt{\left(A^2-(B-1)\right)^2 - 4A^2} \qquad (3.14\text{-}8)$$

D = 1

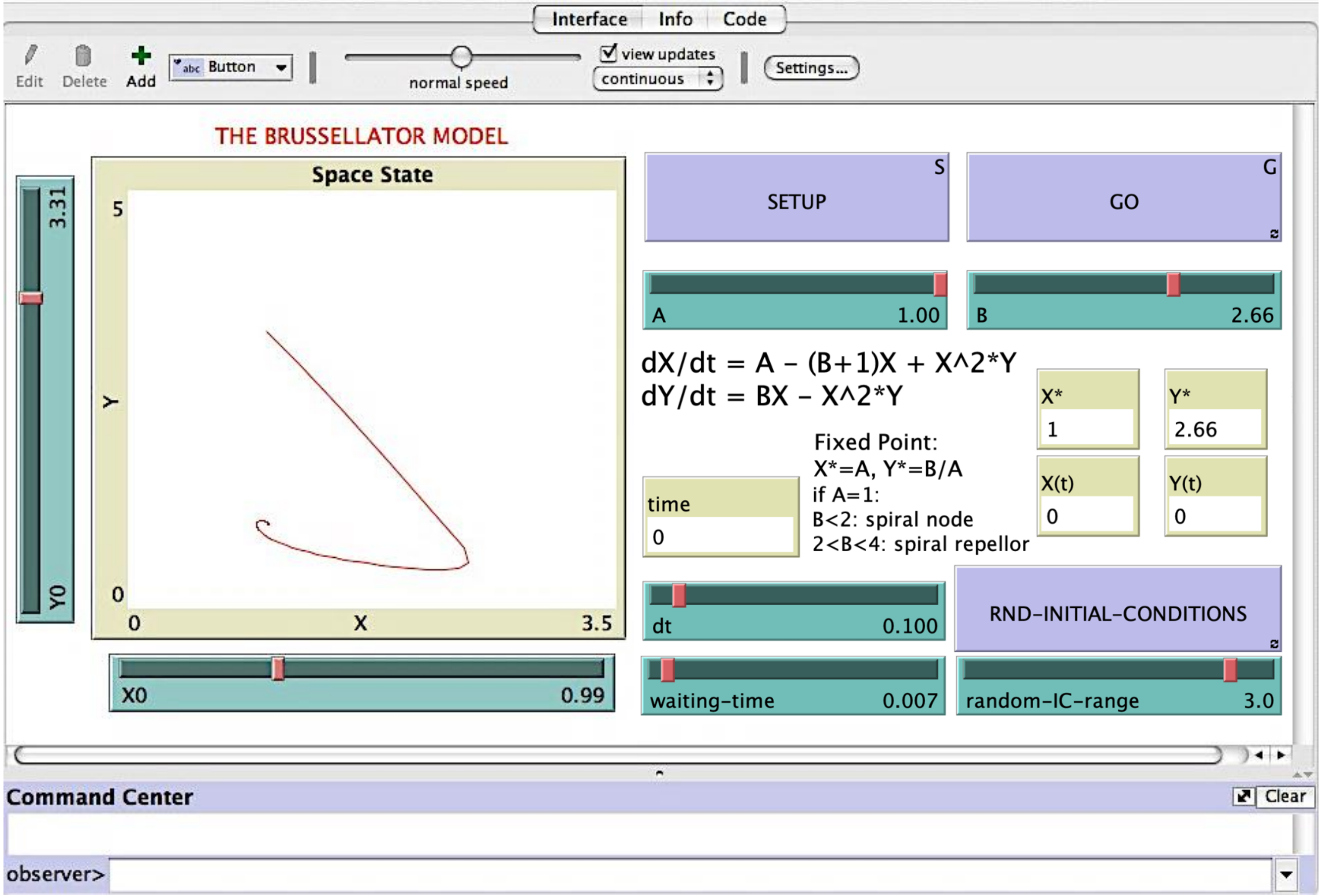
$TrJ = B - 2$

In the discussion of this model, it is traditional to set $\boxed{A = 1}$ and let B be the control parameter. Let us follow that tradition. We see that with $\boxed{B < 2}$, both characteristic values have negative real parts and the fixed point is a spiral node. This result tells us that the chemical concentrations tend toward the fixed point values $X_0 = A = 1$, $Y_0 = B$ as time goes on. They oscillate, however, with the frequency $\Omega = |B(B-4)|^{1/2}$ as they head toward the attractor. For $\boxed{2 < B < 4,}$ the fixed point becomes a spiral repellor. However, our analysis cannot tell us what happens to the trajectories as they spiral away from the fixed point. As we shall learn in the next section, they tend to a limit cycle as shown in Fig. I.1 in Section I (for a different model).

Ex: A=1,B=1 → Δ=1, TrJ=-1, $TrJ^2-4\Delta<0$: Spiral Node (B<2)

A=1,B=3 → Δ=1, TrJ= 1, $TrJ^2-4\Delta<0$: Spiral Repellor (2<B<4)

brussellator.nlogo



Interface
Info
Code
Edit
Delete
Add
Button
Settings...
THE BRUSSELLATOR MODEL
Space State
5
Y
0
0
X
3.5
SETUP
S
GO
G
RND-INITIAL-CONDITIONS
Command Center
Clear
observer>

☑ view updates
continuous
normal speed

A 1.00

B 2.66

Y0 3.31

X0 0.99

dX/dt = A – (B+1)X + X^2*Y
dY/dt = BX – X^2*Y

Fixed Point:
X*=A, Y*=B/A
if A=1:
B<2: spiral node
2<B<4: spiral repellor

X* 1

Y* 2.66

time 0

X(t) 0

Y(t) 0

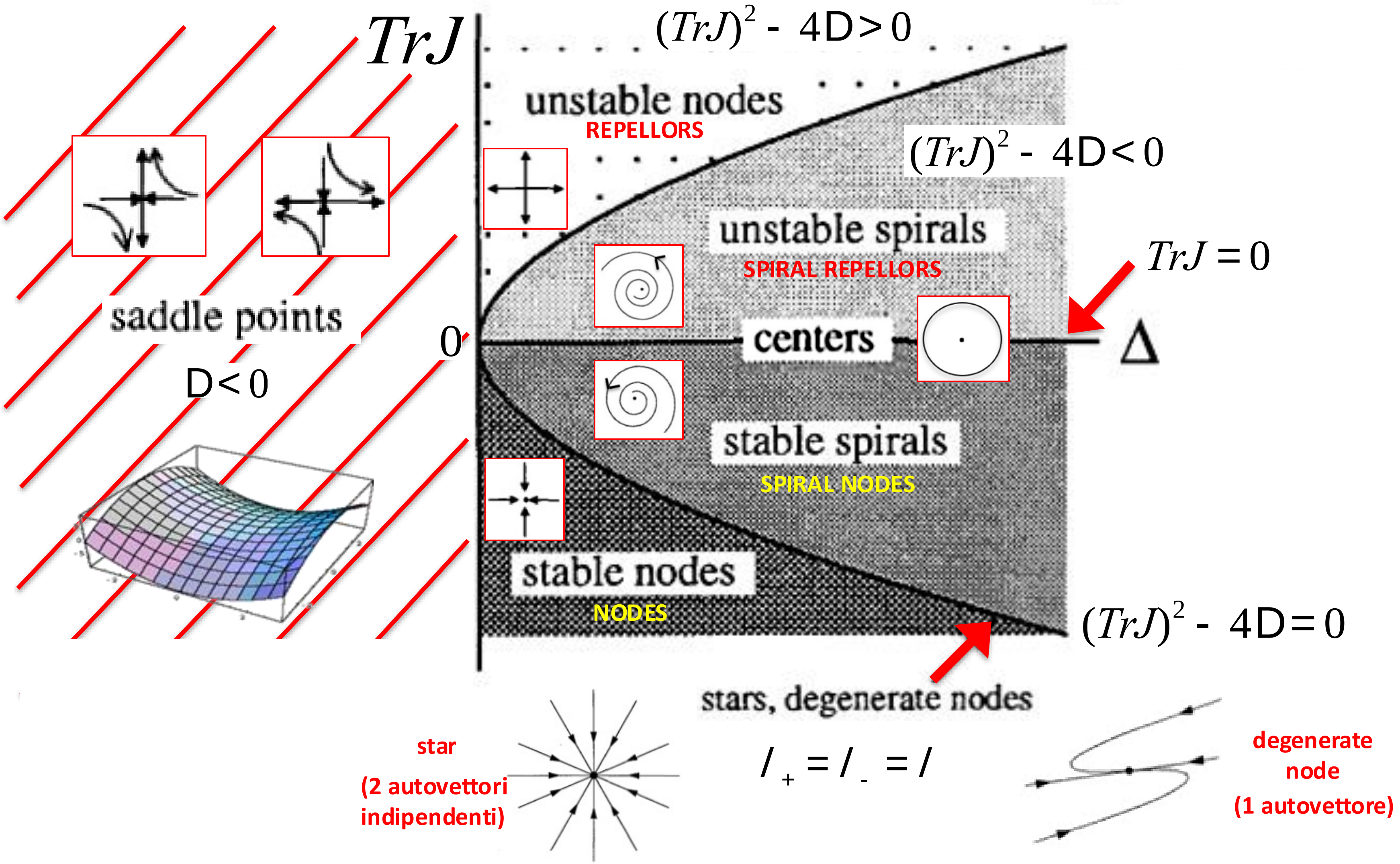
dt 0.100

waiting-time 0.007

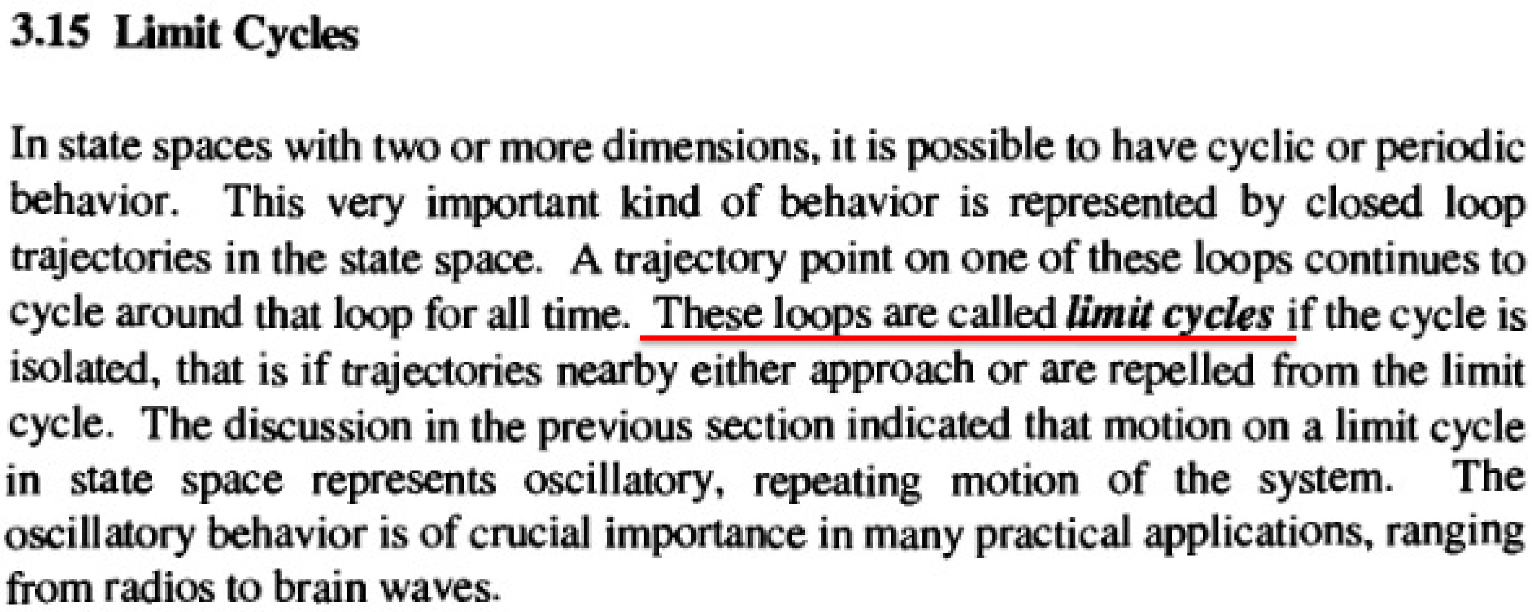
random-IC-range 3.0

Diagramma dei Punti Fissi in uno Spazio degli Stati a Due Dimensioni

$$\lambda_{\pm} = \frac{TrJ \pm \sqrt{(TrJ)^2 - 4\Delta}}{2}$$



TrJ
$(TrJ)^2 - 4\Delta > 0$
unstable nodes
REPELLORS
$(TrJ)^2 - 4\Delta < 0$
unstable spirals
SPIRAL REPELLORS
$TrJ = 0$
saddle points
0
$\Delta < 0$
centers
Δ
stable spirals
SPIRAL NODES
stable nodes
NODES
$(TrJ)^2 - 4\Delta = 0$
stars, degenerate nodes
star
(2 autovettori indipendenti)
$\lambda_{+} = \lambda_{-} = \lambda$
degenerate node
(1 autovettore)



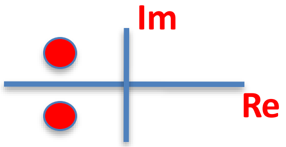
3.15 Limit Cycles

In state spaces with two or more dimensions, it is possible to have cyclic or periodic behavior. This very important kind of behavior is represented by closed loop trajectories in the state space. A trajectory point on one of these loops continues to cycle around that loop for all time. These loops are called ***limit cycles*** if the cycle is isolated, that is if trajectories nearby either approach or are repelled from the limit cycle. The discussion in the previous section indicated that motion on a limit cycle in state space represents oscillatory, repeating motion of the system. The oscillatory behavior is of crucial importance in many practical applications, ranging from radios to brain waves.

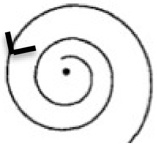
$(TrJ)^2 - 4D < 0$ ® $/_{+}, /_{-}$

complessi coniugati

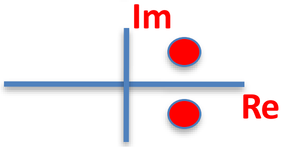
$TrJ < 0$



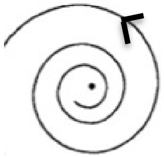
SPIRAL NODE



$TrJ > 0$

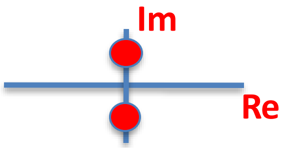


SPIRAL REPELLOR

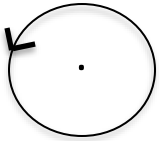


$$x_1(t) = F_1 e^{Rt} \sin Wt$$
$$x_2(t) = F_2 e^{Rt} \sin Wt$$

$R = TrJ = 0$



NASCE UN CICLO LIMITE



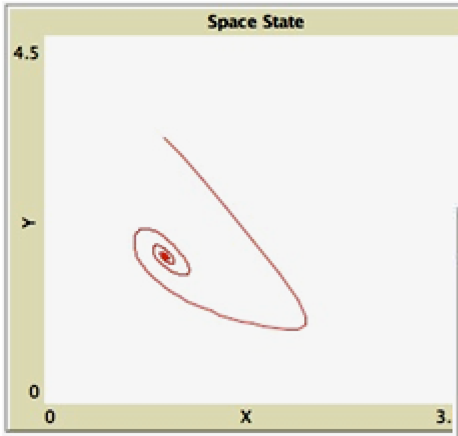
brussellator.nlogo

$$\dot X = A-(B+1)X+X^2Y$$
$$\dot Y = BX - X^2Y$$

1 punto fisso:

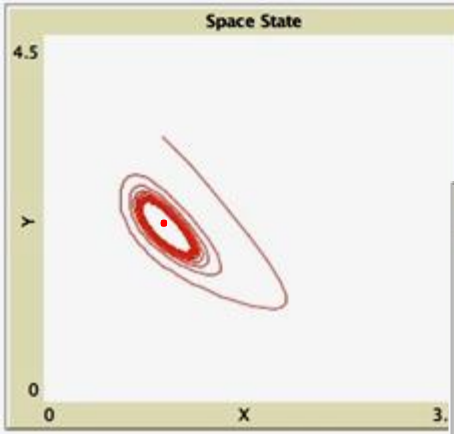
$X_0 = A,\ Y_0 = B/A.$

A=1, B=1.80



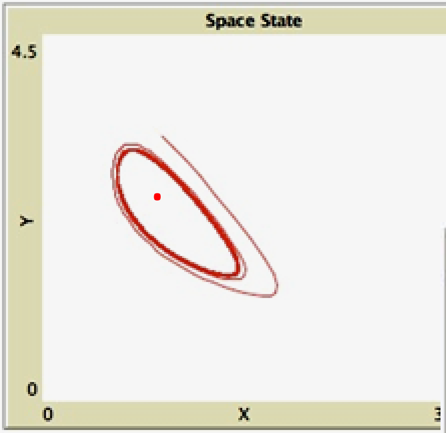
1 stable spiral node

A=1, B=2.15



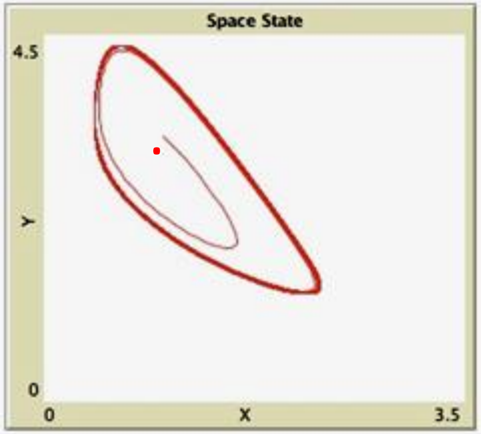
1 unstable spiral repellor
+
1 limit cycle

A=1, B=2.33



1 unstable spiral repellor
+
1 limit cycle

A=1, B=2.85



1 unstable spiral repellor
+
1 limit cycle

A=1,B<2 : stable spiral node

A=1, B=2 : Nasce un ciclo limite!

A=1, B>2 :
unstable spiral repellor + 1 limit cycle